MATH281 200610 Problem Set 12

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Wednesday, April 12, 2006

1. (Based on 8.1.26.) Prove that the general solution of
$$X' = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 4 \\ -6 \end{bmatrix} t + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$
is
$$X = c_1 \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix} e^{\sqrt{2}t} + c_2 \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix} e^{-\sqrt{2}t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

2. (Based on 8.2.2, 8.2.6, and 8.2.8.) Find the general solution of the given system. All differentiation is with respect to the independent variable is t.

   (a) $\begin{aligned} x' &= 2x + 2y \\ y' &= x + 3y \end{aligned}$ (b) $X' = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} X$ (c) $\begin{aligned} x' &= 2x - 7y \\ y' &= 5x + 10y + 4z \\ z' &= 5y + 2z \end{aligned}$

3. (Based on 8.2.10 and 8.2.14.) Solve the following initial value problems.

   (a) $X' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$ (b) $X' = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

4. (Based on 8.4.1, 8.4.2, and 8.4.4.) Use the definition of the matrix exponential to calculate e^{At} and e^{-At} for each of the following matrices. Then use the matrix exponential to find the general solution to the equation $X' = AX$. Finally, use the matrix exponential to find the solution to the initial value problem $X' = AX$, $X(0) = X_0$, where X_0 is some vector of constants.

   (a) $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{bmatrix}$

5. (Based on 8.4.16 and 8.4.18.) Use the Laplace transform (formula (7) in 8.4) to find e^{At} for each of the following matrices. Then use e^{At} to find the general solution to the equation $X' = AX$, and the solution to the initial value problem $X' = AX$, $X(0) = X_0$.

   (a) $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Other problems that will help you learn the material are 8.1.1–26, 8.2.1–14, 8.4.1–8, 8.4.13, and 8.4.15–18. If you are familiar with the concept of diagonalization from your linear algebra course, you can try 8.4.19–24. You should also be able to solve 8.2.19–22, 8.2.29, and 8.2.33–38, and 8.2.46 using the matrix exponential (Laplace transform method); we didn't cover the eigenvalue method, but you could try to figure it out yourself if you're feeling adventurous. The Laplace transform method can also be applied to the 3×3 systems 8.2.23–28 and 8.2.39–45, but the matrix inversion is harder.