

MATH281 200610 Quiz 1 Solutions DRAFT

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1. Because the given function agrees with a polynomial function at any non-zero point on the line, it is differentiable with continuous derivative (i.e., continuously differentiable at such points). To check differentiability at 0 we need to use the definition of derivative: y is differentiable at 0 if and only if the limit

$$l = \lim_{h \rightarrow 0} \frac{y(0+h) - y(0)}{h}$$

exists. To evaluate the limit, we will find it useful to consider left and right-sided limits. From the right,

$$l^+ = \lim_{h \rightarrow 0^+} \frac{y(0+h) - y(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^3 - 0}{h} = \lim_{h \rightarrow 0^+} h = 0,$$

and from the left,

$$l^- = \lim_{h \rightarrow 0^-} \frac{y(0+h) - y(0)}{h} = \lim_{h \rightarrow 0^+} \frac{-h^3 - 0}{h} = \lim_{h \rightarrow 0^+} -h = 0.$$

Since the left and right-sided limits agree, the two-sided limit exists and has the value equal to that of the one-sided limits, i.e., we can conclude that y is differentiable at 0 with derivative $l^+ = l^- = 0$. Therefore the derivative of y is given by

$$y'(x) = \begin{cases} -3x^2, & x < 0 \\ 0, & x = 0 \\ 3x^2, & x > 0. \end{cases}$$

It is readily seen that $y'(x)$ is a continuous function (in fact, $y'(x) = 3|x|x$) so y is continuously differentiable on the entire real line. Substituting what we know about y into the left-hand side L of the differential equation we obtain

$$xy' - 2y = \begin{cases} x(-3x^2) - 3(-x^3) = 0, & x < 0 \\ 0(0) - 3(0) = 0, & x = 0 \\ x(3x^2) - 3(x^3) = 0, & x > 0 \end{cases}$$

i.e., the differential equation is satisfied in all three cases, so we have found a solution defined on the interval $-\infty < x < \infty$.

2. We differentiate the given function to obtain

$$y' = \frac{d}{dx}(1 - ce^{-x})^{-1} = -(1 - ce^{-x})^{-2}(ce^{-x})$$

for the left hand side of the differential equation. For the right hand side we have

$$y - y^2 = \frac{1}{1 - ce^{-x}} - \frac{1}{(1 - ce^{-x})^2} = \frac{1 - ce^{-x} - 1}{(1 - ce^{-x})^2} = \frac{-ce^{-x}}{(1 - ce^{-x})^2}.$$

Since the left hand side equals the right hand side, the given function is a solution to the differential equation on an interval where it is defined and continuous. The interval on which the function is C^1 depends on c ; for example, you might take $I = (\ln c, \infty)$ as the interval (for $c > 0$).

We try to find a value of c which allows the condition $y(-1) = 2$ to be satisfied:

$$\begin{aligned} y(-1) &= 2 \\ \frac{1}{1 - ce^{-(-1)}} &= 2 \\ 1 &= 2 - 2ce^1 \\ 2ce &= 1 \\ c &= \frac{1}{2e}. \end{aligned}$$

Substituting that value of c into the general 'solution' gives the particular 'solution'

$$y(x) = \frac{1}{1 - e^{-x}/(2e)} = \frac{2}{2 - e^{-1-x}}.$$

(Remember that we don't really have a solution until we give an interval I on which the function is defined and C^1 .) We should now verify that the above function is a solution to the initial value problem. Check by identifying an interval I which contains $x = -1$ and on which the function is defined and C^1 ($I = (-\ln(2) - 1, \infty)$ works, for example), then by taking a derivative and checking that the left-hand and right-hand sides of the equation are identically equal for all x , and finally by checking that the initial value condition is satisfied.

3. (a) Following the technique of separation of variables we have

$$\begin{aligned} \frac{dy}{y - y^2} &= dx \\ \int \frac{dy}{y - y^2} &= \int dx = x + C. \end{aligned}$$

Expanding the integrand above using partial fractions,

$$\frac{1}{y - y^2} = \frac{1}{y(1 - y)} = \frac{A}{y} + \frac{B}{1 - y} = \frac{A(1 - y) + By}{y(1 - y)} = \frac{A + (-A + B)y}{y(1 - y)}.$$

Comparing numerators in the above tells us that $A = 1$ and $B = 1$ so we have

$$\int \frac{dy}{y - y^2} = \int \left(\frac{1}{y} + \frac{1}{1 - y} \right) dy = \ln|y| - \ln|1 - y| = \ln \left| \frac{y}{1 - y} \right|.$$

Therefore a solution to the equation is given in implicit form by

$$\ln \left| \frac{y}{1 - y} \right| = x + C.$$

Solving for y ,

$$\begin{aligned} \left| \frac{y}{1 - y} \right| &= e^{x+C} = ke^x, \quad k > 0 \\ \frac{y}{1 - y} &= \pm ke^x, \quad k > 0 \end{aligned}$$

The notation is simplified if we drop the \pm and allow k to be either positive or negative (but not 0).

$$\begin{aligned}\frac{y}{1-y} &= ke^x \\ y &= ke^x(1-y) \\ (1+ke^x)y &= ke^x \\ y &= \frac{ke^x}{1+ke^x} = \frac{1}{(1+ke^x)/(ke^x)} = \frac{1}{(1/k)e^{-x} + 1}\end{aligned}$$

which agrees with the given solution (with $c = -1/k$).

Two solutions are potentially lost: the solutions where the right hand side is zero, which are therefore constant solutions, which therefore must be $y = 0$ or $y = 1$. The solution $y = 1$ can be recovered from the given formula by setting $c = 0$. The solution $y = 0$ cannot be recovered by setting c equal to anything, so is 'lost'. (It can be recovered by setting $k = 0$, i.e., $c = \infty$, though.)

- (b) There are three qualitatively different situations for the formula given for the solution to the differential equation: $c > 0$, $c = 0$, $c < 0$, and there is also the 'lost' solution $y = 0$. If $c > 0$ we can write

$$y = \frac{1}{1 + e^{-x + \ln c}}$$

which is of the form $\phi_1(x - l)$ where $l = \ln c$ and

$$\phi_1(x) = \frac{1}{1 + e^{-x}},$$

i.e., all solutions where $c > 0$ can be obtained by translating $\phi_1(x)$ along the x -axis. Similarly, all solutions where $c < 0$ can be obtained by translating

$$\phi_2(x) = \frac{1}{1 - e^{-x}}$$

along the x -axis. Translating the equilibrium solution $y = 1$ along the x -axis doesn't give anything new; the same goes for the 'lost' equilibrium solution $y = 0$. Graphing a few solution curves in each case, we have Figure 1 in which you can see four curves for $c < 0$ in blue, the one curve for $c = 0$ in purple, five curves for $c > 0$ in red, and the one 'lost' curve (you might say $c = \infty$ for this extreme curve) in green.

Note that the function $\phi_2(x)$ actually gives us two qualitatively different solutions to the differential equation, one where $x > 0$ and one where $x < 0$.

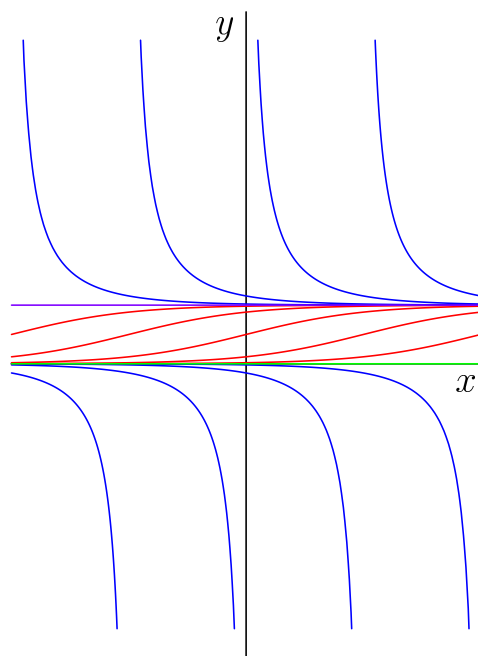


Figure 1: Solution curves for $y' = y - y^2$