MATH281 200610 Quiz 2 Solutions DRAFT

Edward Doolittle

Wednesday, February 1, 2006

1. Separating variables and integrating by partial fractions as in the lectures,

$$\frac{dy}{y^2-1} = \frac{dx}{x^2-1}$$
$$\frac{1}{2}\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy = \frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$$
$$\ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right| + c.$$

Exponentiating both sides and dropping the absolute value signs,

$$\frac{y-1}{y+1} = \pm k\frac{x-1}{x+1},$$

where $k > 0$. The positive parameter k can be replaced by a parameter k with no restrictions (careful about the case $k = 0$ which returns the (until now) lost solution $y = 1$) giving the general solution

$$\frac{y-1}{y+1} = k\frac{x-1}{x+1}.$$

We could now continue to solve for y to find an explicit general solution, but we can save ourselves a bit of headache by substituting the initial condition into the implicit general solution and then finding an explicit particular solution. Setting $x = 2$ and $y = 2$ and then solving for k,

$$\frac{1}{3} = k\frac{1}{3}$$
$$k = 1.$$

Now substituting that value of k into the general solution and then solving for y gives the particular solution $y = x$. You should check that that is indeed a solution to the initial value problem; in particular, you should identify an interval on which the solution makes sense; the interval shouldn't include $x = 1$ (why not?).

2. Separating variables,

$$\frac{\cos y}{\sin^2 y}\,dy = \frac{2x}{x^2+10}\,dx$$
$$\int \cot y \csc y\,dy = \int \frac{2x}{x^2+10}\,dx.$$

The left hand integral can be done by the hint. The right hand integral can be done by making the substitution $u = x^2 + 10$, $du = 2x\,dx$:

$$-\csc y = \int \frac{1}{u}\,du = \ln|u| - c = \ln|x^2+10| - c = \ln(x^2+10) - c$$

which gives the implicit solution

$$\csc y + \ln(x^2 + 10) = c,$$

where I chose $-c$ as the constant of integration above just so that I would get no minus signs in the final answer, but of course a minus sign is OK.

Lost solutions will appear when we might be dividing by 0, i.e., when $x^2 + 10 = 0$ (never) or when $\sin y = 0$ (which gives the lost solutions $y = k\pi$, k an integer). None of the latter can be found from the implicit solution because y in the implicit solution always varies as x varies.

(Note: I suggest leaving the solution in implicit form, and not bothering with a discussion of the interval on which the solution makes sense, but it is possible to give a solution in explicit form by solving for y. See below for a discussion which I have included just so you can see what you might have to contend with in order to solve for y explicitly. The situation is in general hopeless when both x and y appears in more than one place in an equation.)

3. This is just a slightly more complicated example of partial fractions than what we have studied so far. Separating variables,

$$\frac{dy}{y-y^3} = dx$$
$$\int \frac{dy}{y-y^3} = \int dx = x + c.$$

To do the left-hand integral above, factor the denominator to obtain

$$\frac{1}{y-y^3} = \frac{1}{y(1-y)(1+y)}.$$

Now we look for constants A, B, and C which make the equation

$$\frac{1}{y(1-y)(1+y)} = \frac{A}{y} + \frac{B}{1-y} + \frac{C}{1+y}$$

an identity (true for all y). Gather the right-hand side by finding a common denominator:

$$\frac{1}{y(1-y)(1+y)} = \frac{A(1-y)(1+y) + By(1+y) + Cy(1-y)}{y(1-y)(1+y)}.$$

For the above to be an identity we must have

$$1 = A(1-y)(1+y) + By(1+y) + Cy(1-y)$$

for all y. Now there are two ways to proceed: we could expand the right-hand side of the above, set the constant coefficient equal to 1, set the coefficient of y equal to 0, and set the coefficient of y^2 equal to 0, or we can try a trick to make the calculations easier. Setting $y = 0$ in the above, we have

$$1 = A(1-0)(1+0) + B(0)(1+0) + C(0)(1-0)$$

which show that $A = 1$. Similarly, setting $y = 1$ gives $B = 1/2$ and setting $y = -1$ gives $C = -1/2$. So we have

$$\int \frac{dy}{y-y^3} = \int \left(\frac{1}{y} + \frac{1}{2}\frac{1}{1-y} - \frac{1}{2}\frac{1}{1+y} \right) dy = \ln|y| - \frac{1}{2}\ln|1-y| - \frac{1}{2}\ln|1+y|.$$

Substituting that into the solution to the differential equation and using laws of logarithms gives

$$\ln\left|\frac{y}{\sqrt{1-y^2}}\right| = x + c$$

or, exponentiating both sides, etc.,

$$\frac{y}{\sqrt{1-y^2}} = ke^x.$$

Fortunately, it is not too difficult to solve for y in this case to obtain the explicit solution

$$y = \pm\sqrt{\frac{k^2e^{2x}}{1+k^2e^{2x}}}.$$

You should check that the above really is a solution, and give an interval on which the solution makes sense.

2. (Continued) It is possible to find an explicit formula for y in terms of x for this problem, but it is rather difficult to unravel all the branches of the inverse functions that are involved. Branches occur because the cosecant function is not invertible. We actually get a whole mess of solutions in two families (much like the plus/minus families you get when taking a square root), each family indexed by an integer k:

$$y = 2k\pi + \csc^{-1}(c - \ln(x^2+10)),$$
$$y = (2k+1)\pi - \csc^{-1}(c - \ln(x^2+10)).$$

The $2k\pi$ term appears because cosecant is periodic with period 2π. In order to understand what is happening here, I strongly suggest you graph the function $z = \csc y$ and solve the equation $\csc y = A$ using the graph.

The situation is further complicated because the above functions are not defined for all x for some values of c. For $c < \ln(10) - 1$, the functions are defined, and give solutions to the differential equation, for all x, which appear in Figure 1 in red. For c greater the critical value of $\ln(10) - 1$ and less than the critical value of $\ln(10) + 1$, the solutions 'bifurcate' into two separate curves, which appear in blue and green in Figure 1. For c greater than the critical value of $\ln(10) + 1$ the solutions again break into three parts which appear in purple: two parts nested inside the blue and green curves, and a third completely new part also in purple which appears in the as-yet vacant strips in the plane. Finally, the equilibrium solutions $y = k\pi$ split the plane into strips, and appear in Figure 1 as horizontal black lines. No solution is possible when $y = k\pi + \pi/2$; why not?

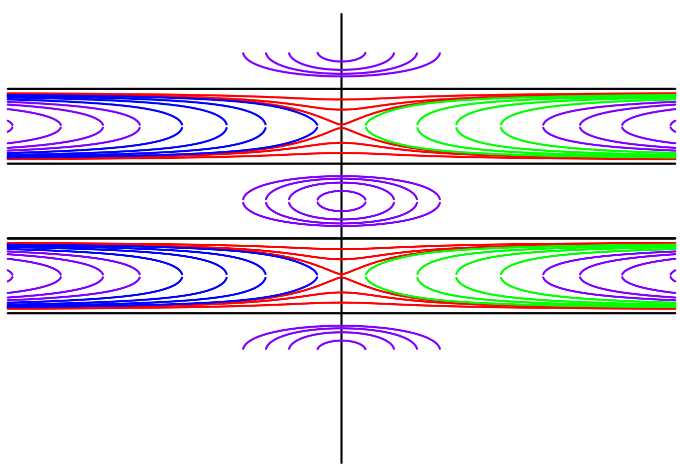

Figure 1: Solution curves satisfying $\csc y + \ln(x^2+10) = c$