

# MATH281 200610 Quiz 3 Solutions DRAFT

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1. Taking derivatives of  $y_1 = \cos(\ln x)$ ,

$$\begin{aligned}y_1' &= -\sin(\ln x) \frac{1}{x} \\y_1'' &= -\cos(\ln x) \frac{1}{x^2} + \sin(\ln x) \frac{1}{x^2},\end{aligned}$$

and substituting into the differential equation,

$$Ly_1 = x^2 y_1'' + x y_1' + y_1 = -\cos(\ln x) + \sin(\ln x) - \sin(\ln x) + \cos(\ln x) = 0,$$

so  $y_1$  satisfies the differential equation. Similarly, taking derivatives of  $y_2 = \sin(\ln x)$ ,

$$\begin{aligned}y_2' &= \cos(\ln x) \frac{1}{x} \\y_2'' &= -\sin(\ln x) \frac{1}{x^2} - \cos(\ln x) \frac{1}{x^2},\end{aligned}$$

and substituting into the differential equation,

$$Ly_2 = x^2 y_2'' + x y_2' + y_2 = -\sin(\ln x) - \cos(\ln x) + \cos(\ln x) + \sin(\ln x) = 0,$$

so  $y_2$  also satisfies the differential equation. Taking the Wronskian,

$$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\sin(\ln x) \frac{1}{x} & \cos(\ln x) \frac{1}{x} \end{vmatrix} = \frac{1}{x} (\cos^2(\ln x) + \sin^2(\ln x)) = \frac{1}{x}$$

which is never zero on  $I$ , so the given functions are linearly independent and therefore form a fundamental system of solutions. The general solution is  $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$ .

2. The function  $y_1(x) = x^2$  is a solution to the differential equation on  $I = (0, \infty)$  because  $x^2(2) + 2x(2x) - 6(x^2) = 2x^2 + 4x^2 - 6x^2 = 0$ . For reduction of order, let  $y_2(x) = u(x)y_1(x) = u(x)x^2$ ; taking derivatives,

$$\begin{aligned}y_2' &= u'x^2 + 2ux \\y_2'' &= u''x^2 + 4u'x + 2u.\end{aligned}$$

Substituting into the differential equation,

$$\begin{aligned}Ly_2 &= x^2 y_2'' + 2x y_2' - 6y_2 \\&= x^2(u''x^2 + 4u'x + 2u) + 2x(u'x^2 + 2ux) - 6ux^2 \\&= x^4 u'' + 6x^3 u' + ux^2(2 + 4 - 6) \\&= x^4 u'' + 6x^3 u' = 0.\end{aligned}$$

Setting  $v = u'$ , we have the first order linear differential equation  $xv' + 6v = 0$ . The equation can be solved using the techniques of chapter 2.3. In detail, divide the equation through by  $x$  to obtain

$v' + (6/x)v = 0$ . Then an integrating factor is  $\exp(-\int 6/x dx) = \exp(-6 \ln x) = x^{-6}$ . Multiplying the equation  $v' + (6/x)v = 0$  by the integrating factor  $x^6$ , we obtain  $(x^6 v)' = 0$  with a solution  $v = x^{-6}$ . (We only need one solution because we're looking for one solution to  $Ly_2 = 0$ .) We then obtain  $u = x^{-5}$  times some constant that can be ignored. Then  $y_2 = uy_1 = x^{-5}x^2 = x^{-3}$ . (It is a good idea to double check at this point that  $y_2(x) = x^{-3}$  really is a solution.) The pair of solutions  $y_1, y_2$  is linearly independent because

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^{-3} \\ 2x & -3x^{-4} \end{vmatrix} = -3x^{-2} + 2x^{-2} = -x^{-2} \neq 0 \text{ on } I.$$

It follows that  $y_1, y_2$  is a fundamental set of solutions to the differential equation on  $I$ , so the general solution on  $I$  is  $y = c_1 x^2 + c_2 x^{-3}$ .