MATH281 200610 Quiz 4 Solutions DRAFT

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1. Consider solutions of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$
$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Multiplying the first derivative by $2x$ and the second derivative by x^2 gives

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^n$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$
$$2xy' = \sum_{n=1}^{\infty} 2n c_n x^n.$$

Shifting all the series so that they are in terms of x^k (the series for x^2y'' and $2xy'$ don't have to change; for the series for y'' we make the change $k = n-2$, $n = k+2$) we have

$$x^2 y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^k$$
$$y'' = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k$$
$$2xy' = \sum_{k=1}^{\infty} 2k c_k x^k.$$

In order to add the series, we not only have to have the powers of x in the terms equal, we also should have the starting indices equal. There are two ways to do that. The first way, which always works, is to choose the highest starting index that appears in the series and treat the other terms of some of the series as exceptional. Using that method we have

$$x^2 y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^k$$
$$y'' = 2c_2 + 6c_3 x + \sum_{k=2}^{\infty} (k+2)(k+1) c_{k+2} x^k$$
$$2xy' = 2c_1 x + \sum_{k=2}^{\infty} 2k c_k x^k$$

and adding gives

$$(x^2+1)y'' + 2xy' = 2c_2 + (2c_1+6c_3)x + \sum_{k=2}^{\infty} \left(k(k-1)c_k + (k+1)(k+2)c_{k+2} + 2kc_k\right) x^k.$$

The second, trickier method, which only works sometimes, is to note that we can re-write some of the series with lower starting indices because the extra terms added in that manner are zero. In particular,

$$\begin{aligned} x^2y'' &= \sum_{k=0}^{\infty} k(k-1)c_k x^k \\ y'' &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k \\ 2xy' &= \sum_{k=0}^{\infty} 2kc_k x^k \end{aligned}$$

because all the added terms are zero. Adding, we have

$$(x^2+1)y'' + 2xy' = \sum_{k=0}^{\infty} \left(k(k-1)c_k + (k+1)(k+2)c_{k+2} + 2kc_k\right) x^k$$

which is the same result as the previous, but more compactly expressed. Rearranging, we have

$$(x^2+1)y'' + 2xy' = \sum_{k=0}^{\infty} \left(k(k+1)c_k + (k+1)(k+2)c_{k+2}\right) x^k$$

so in order for the differential equation to be satisfied we must have

$$k(k+1)c_k + (k+1)(k+2)c_{k+2} = 0$$

for all $k = 0, 1, 2, \ldots$. Dividing through by $k+1$ which is never zero we have

$$kc_k + (k+2)c_{k+2} = 0$$

or, rearranging, we have the recursion relation

$$c_{k+2} = -\frac{k}{k+2}c_k.$$

Now, for the initial conditions to be satisfied, we must have

$$\begin{aligned} 0 = y(0) &= c_0 + c_1(0) + c_2(0)^2 + \cdots = c_0 \\ 1 = y'(0) &= c_1 + 2c_2(0) + 3c_3(0)^2 + \cdots = c_1. \end{aligned}$$

Applying the recursion relation gives $c_2 = c_4 = c_6 = \cdots = 0$, $c_3 = -1/3$, $c_5 = 1/5$, $c_7 = -1/7$, etc. In general we have $c_{2m} = 0$ and $c_{2m+1} = (-1)^m/(2m+1)$, so the particular solution to the given initial value problem is

$$y = \frac{1}{1}x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots.$$

You should check that the above function satisfies the initial value problem. (You may also be able to find a closed-form representation of y and check that that function satisfies the IVP: consider the series for logarithm as a starting point.)

2. The function in question is

$$f(t) = \begin{cases} a(b-t) & t < 1 \\ 0 & t \geq 1 \end{cases}$$

The Laplace transform of f is

$$\mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^b e^{-st} f(t)\, dt$$

because f vanishes for $t > b$. Filling in what we know of f,

$$\mathscr{L}\{f\}(s) = a \int_0^b e^{-st}(b-t)\, dt = ab \int_0^b e^{-st}\, dt - a \int_0^b e^{-st} t\, dt.$$

The second last integral above is straightforward to evaluate:

$$\int_0^b e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^b = \frac{1}{s} - \frac{e^{-bs}}{s}.$$

The last integral above should be evaluated by parts. Let $u = t$, $dv = e^{-st}\, dt$, $du = dt$, $v = -e^{-st}/s$. Then

$$\int_0^b e^{-st} t\, dt = -\frac{t}{s} e^{-st}\Big|_0^b + \int_0^b \frac{e^{-st}}{s}\, dt = -\frac{t}{s} e^{-st}\Big|_0^b - \frac{e^{-st}}{s^2}\Big|_0^b.$$

(Check the indefinite integral by differentiating.) Substituting the end points $t = 0$ and $t = b$ into the above gives

$$\int_0^b e^{-st} t\, dt = -\frac{b}{s} e^{-bs} - \frac{e^{-bs}}{s^2} + \frac{1}{s^2}.$$

Gathering all of the above results together,

$$\mathscr{L}\{f\}(s) = ab\left(\frac{1}{s} - \frac{e^{-bs}}{s}\right) - a\left(-\frac{b}{s} e^{-bs} - \frac{e^{-bs}}{s^2} + \frac{1}{s^2}\right) = \frac{ab}{s} + (e^{-bs} - 1)\frac{a}{s^2}.$$

How could you use the Heaviside step function to check your results?