MATH281 200610 Sample Final 1 Solutions DRAFT

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1. Using one of the laws of exponents the equation can be written as
$$\frac{dy}{dx} = e^y e^x$$
which is separable. Moving all the ys to one side and all the xs to the other,
$$e^{-y}\,dy = e^x dx.$$
Integrating,
$$-e^{-y} = e^x + C.$$
Solving for y,
$$y = -\ln(-e^x - C).$$
The domain of a solution depends on C; in detail, we must have $-e^x - C > 0$ which implies $e^x < -C$ or $x < \ln(-C)$.

2. Here, write
$$\begin{aligned} M(x,y) &= 4x + y \\ N(x,y) &= 4y + x \end{aligned}$$
and note that
$$\frac{\partial}{\partial y} M(x,y) = 1 = \frac{\partial}{\partial x} N(x,y)$$
so the equation is exact. We look for a function $f(x,y)$ with the property that
$$\frac{\partial f}{\partial x} = M(x,y) = 4x + y$$
which implies that
$$f(x,y) = 2x^2 + xy + g(y).$$
Differentiating with respect to y we get
$$\frac{\partial f}{\partial y} = N(x,y) = x + g'(y)$$
which implies that $g'(y) = 4y$, or $g(y) = 2y^2$. Altogether, we have $M(x,y)dx + N(x,y)dy = df(x,y)$ where $f(x,y) = 2x^2 + xy + 2y^2$, so the general solution is
$$2x^2 + xy + 2y^2 = c.$$
For the curve to pass through $(-2,1)$ we must have
$$c = 2(-2)^2 + (-2)(1) + 2(1)^2 = 8.$$
In summary, the required curve has equation
$$2x^2 + xy + 2y^2 = 8.$$

3. It might be a good idea to check that the given function really is a solution. Taking derivatives, we have

$$\begin{aligned} y &= \cos(x^2) \\ y' &= -2x\sin(x^2) \\ y'' &= -2\sin(x^2) - 4x^2\cos(x^2). \end{aligned}$$

Applying the linear operator we have

$$L(y) = xy'' - y' + 4x^3 y = x(-2\sin(x^2) - 4x^2\cos(x^2) + 2x\sin(x^2) + 4x^3\cos(x^2) = 0,$$

so y is a solution to the differential equation. To find another solution, we use variation of parameters. Let $y_1(x) = u(x)\cos(x^2)$; then we have

$$\begin{aligned} y_1 &= u(x)\cos(x^2) \\ y_1' &= u'(x)\cos(x^2) + u(-2x\sin(x^2)) \\ y_1'' &= u''(x)\cos(x^2) + 2u'(-2x\sin(x^2)) + u(-2\sin(x^2) - 4x^2\cos(x^2)). \end{aligned}$$

Applying the linear operator,

$$\begin{aligned} L(y_1) &= xu''\cos(x^2) + 2xu'(-2x\sin(x^2)) + xu(-2\sin(x^2) - 4x^2\cos(x^2)) - u'\cos(x^2) - u(-2x\sin(x^2)) + 4x^3u\cos(x^2). \\ &= (x\cos(x^2))u'' + (-4x^2\sin(x^2) - \cos(x^2))u'. \end{aligned}$$

Let $v = u'$. Then we have a linear first order equation

$$x\cos(x^2)v' + (-4x^2\sin(x^2) - \cos(x^2))v = 0.$$

Putting it in standard form by dividing through by the coefficient of v',

$$v' + \left(-4x\tan(x^2) - \frac{1}{x}\right)v = 0.$$

An integrating factor is

$$e^{\int -4x\tan(x^2) - (1/x)\,dx} = e^{2\ln|sec(x^2)| + \ln x} = (x\sec^2(x^2))^{-1}.$$

Checking,

$$\begin{aligned} &\frac{d}{dx}((x\sec^2(x^2))^{-1}v) \\ &\qquad = (x\sec^2(x^2))^{-1}v' - (x\sec^2(x^2))^{-2}(\sec^2(x^2) + 2x\sec(x^2)\sec(x^2)\tan(x^2)2x)v \\ &\qquad\qquad = (x\sec^2(x^2))^{-1}(v' - ((1/x) - 4x\tan(x^2))v). \end{aligned}$$

It follows that

$$v = kx\sec^2(x^2).$$

We can take $k = 2$ because we only need one solution. Integrating again,

$$u = \tan(x^2)$$

so $y_1 = \tan(x^2)\cos(x^2) = \sin(x^2)$ is another solution. You should check that $\cos(x^2)$ and $\sin(x^2)$ are linearly independent by showing that the Wronsikian $W(\cos(x^2), \sin(x^2))$ is never zero.

4. First, we need to solve the homogeneous equation $y'' + y = 0$. You should remember that a fundamental system of solutions for this equation is $\cos x$, $\sin x$. (If you don't remember that, you can find the

auxiliary equation, etc.) Now by variation of parameters, a particular solution to the homogeneous equation is given by $y = c_1(x)\cos x + c_2(x)\sin x$ where

$$\begin{aligned} c_1(x) &= \int \frac{\Delta_1}{\Delta}\, dx \\ c_2(x) &= \int \frac{\Delta_2}{\Delta}\, dx, \end{aligned}$$

where

$$\begin{aligned} \Delta &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \\ \Delta_1 &= \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sec x \sin x = -\tan x \\ \Delta_2 &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \sin x. \end{aligned}$$

Filling in those results,

$$\begin{aligned} c_1(x) &= \int -\tan x\; dx = \ln|\cos x|, \\ c_2(x) &= \int \sin x\; dx = -\cos x, \end{aligned}$$

so in summary the general solution to the differential equation is

$$y = c_1 \cos x + c_2 \sin x + \ln|\cos x|\cos x - \cos x \sin x.$$

You should check the answer.

5. As usual, we have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} \end{aligned}$$

The initial condition $y(0) = 4$ implies $c_0 = 4$. Then the differential operator is

$$xy' + 2y' - 2y = \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=1}^{\infty} 2n c_n x^{n-1} + \sum_{n=0}^{\infty} (-2c_n)x^n.$$

Making the change of index $k = n - 1$ in the second sum, and writing $k = n$ in the other two sums, we have

$$xy' + 2y' - 2y = \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} 2(k+1)c_{k+1} x^k + \sum_{k=0}^{\infty} (-2c_k)x^k.$$

Now, changing the index in the first sum (which is possible because the term that is introduced is just 0), we can write

$$xy' + 2y' - 2y = \sum_{k=0}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} 2(k+1)c_{k+1} x^k + \sum_{k=0}^{\infty} (-2c_k)x^k = \sum_{k=0}^{\infty} (k c_k + 2(k+1)c_{k+1} - 2c_k)x^k.$$

In order to solve the differential equation we must have

$$(k-2)c_k + 2(k+1)c_{k+1} = 0 \implies c_{k+1} = \frac{2-k}{2k+2}c_k$$

for all $k = 0, 1, 2, \ldots$. The first few c_k can be written

$$\begin{aligned} c_1 &= \frac{2-0}{2(0+2}c_0 = c_0 = 4 \\ c_2 &= \frac{2-1}{2(1)+2}c_1 = \frac{1}{4}c_1 = 1 \\ c_3 &= \frac{2-2}{2(2)+2}c_2 = 0 \\ c_4 &= \frac{2-3}{2(3)+2}c_3 = 0 \end{aligned}$$

and so on; most of the coefficients are 0, and we have $y(x) = 4 + 4x + x^2$ as the solution. (Check!)

6. First, we use the formula for the Laplace transform of a derivative to obtain

$$\mathscr{L}\{te^{3t}\cos t\}(s) = -\frac{d}{ds}\mathscr{L}\{e^{3t}\cos t\}(s).$$

Then we use the formula for translation in the s-axis to obtain

$$\mathscr{L}\{e^{3t}\cos t\}(s) = \mathscr{L}\{\cos t\}(s-3) = \frac{s-3}{(s-3)^2+1}.$$

Reversing our steps,

$$\mathscr{L}\{te^{3t}\cos t\}(s) = -\frac{d}{ds}\frac{s-3}{(s-3)^2+1} = \frac{(s-3)2(s-3) - ((s-3)^2+1)}{((s-3)^2+1)^2} = \frac{(s-3)^2-1}{((s-3)^2+1)^2}.$$

7. Taking the Laplace transform of both sides of the equation,

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 3Y(s) = \frac{3}{s^2+9}.$$

Using the initial conditions $y(0) = 0$, $y'(0) = 0$ and solving for $Y(s)$,

$$Y(s) = \frac{3}{(s^2+9)(s^2+4s+3)}.$$

Factoring the denominator and applying partial fractions,

$$Y(s) = \frac{As+B}{s^2+9} + \frac{C}{s+1} + \frac{D}{s+3} = \frac{(s+1)(s+3)(As+B) + (s+3)(s^2+9)C + (s+1)(s^2+9)D}{(s^2+9)(s+1)(s+3)}.$$

Setting $s = -1$ we have $C = 3/20$. Setting $s = -3$ gives $D = -1/12$. Setting $s = 0$ gives $3B + 81/20 - 9/12 = 3$ so $B = -5/60$. Finally, comparing coefficients of s^3 gives $A + C + D = 0$ so $A = 2/60$. In summary,

$$Y(s) = \frac{2}{60}\frac{s}{s^2+9} - \frac{5}{60}\frac{1}{s^2+9} + \frac{9}{60}\frac{1}{s+1} - \frac{45}{60}\frac{1}{s+3}.$$

Taking the inverse Laplace transform,

$$y(t) = \frac{2}{60}\cos 3t - \frac{5}{20}\sin 3t + \frac{9}{60}e^{-t} - \frac{45}{60}e^{-3t}.$$

If you check the answer, you'll find that it is wrong. Now go back and fix the mistakes.

8. We can write the system as

$$X' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} X.$$

The characteristic polynomial is $(1-\lambda)(1-\lambda) - (2)(-2) = \lambda^2 - 2\lambda + 5$ which has complex roots, so if you were to use the eigenvalue method, you would have to contend with complex eigenvalues and complex eigenvectors, which we did not study. Instead, let's find the matrix exponential using the inverse Laplace transform. We calculate

$$sI - A = \begin{bmatrix} s-1 & -2 \\ 2 & s-1 \end{bmatrix};$$

taking the inverse,

$$(sI-A)^{-1} = \frac{1}{(s-1)^2+4} \begin{bmatrix} s-1 & 2 \\ -2 & s-1 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-1)^2+4} & \frac{2}{(s-1)^2+4} \\ \frac{-2}{(s-1)^2+4} & \frac{s-1}{(s-1)^2+4} \end{bmatrix}.$$

Taking the inverse Laplace transform using translation in the s-axis,

$$\mathscr{L}^{-1}\left\{ \begin{bmatrix} \frac{s-1}{(s-1)^2+4} & \frac{2}{(s-1)^2+4} \\ \frac{-2}{(s-1)^2+4} & \frac{s-1}{(s-1)^2+4} \end{bmatrix} \right\} = \begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ -e^t \sin 2t & e^t \cos 2t \end{bmatrix}.$$

You should check that the matrix is a fundamental matrix for the equation, i.e., that $\Phi'(t) = A\Phi(t)$ and that $\det(\Phi(t)) \neq 0$. The columns of a fundamental matrix give linearly independent solutions, so the final answer to the question is

$$X_1 = \begin{bmatrix} e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix}, \quad X_2 = \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}.$$

Check! I haven't checked it carefully, so there may be errors.

9. This problem was solved in problem set 12. Alternatively, you could find the matrix exponential using the series; the work is simplified because $A^2 = I$, i.e., the matrix is idempotent. Or you could use eigenvalues and eigenvectors, which is complicated by the multiple eigenvalue $\lambda = 1$; we didn't study the theory of multiple eigenvalues, but you could use it if you want. Or you could decouple the system into

$$x_2' = x_2$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

and solve the two decoupled systems separately. The second system doesn't have any repeated eigenvalues, so it can be handled by the methods of section 8.2.1. All those methods are ad hoc or fussy and complicated, though; the best way to solve the problem is using the method of problem set 12.