MATH281 200610 Sample Final 2 Solutions DRAFT

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1. The equation is linear, so in order to solve it we solve the homogeneous equation first. The homogeneous equation (in standard form) is
$$y' + \frac{3}{x}y = 0.$$
You can use separation of variables, or you can find an integrating factor by other means (memorizing the formula, guessing, etc.). Separating variables,
$$\frac{dy}{y} = -3\frac{dx}{x} \implies \ln|y| = -3\ln|x| + C \implies y = \frac{k}{x^3},$$
so an integrating factor is x^3. Multiplying the given equation (in standard form) by the integrating factor we have
$$y' + \frac{3}{x}y = x^2 - 2x + 4 \implies x^3y' + 3x^2y = (x^3y)' = x^5 - 2x^4 + 4x^3.$$
Integrating,
$$x^3y = \frac{1}{6}x^6 - \frac{2}{5}x^5 + x^4 + C \implies y = \frac{1}{6}x^3 - \frac{2}{5}x^2 + x + Cx^{-3}$$
is the general solution. Check!

2. (a) We have
$$y_1' = 2x$$
$$y_1'' = 2$$
so
$$x^2y_1'' - 2y_1 = x^2(2) - 2x^2 = 0$$
so y_1 is a solution. Similarly, on the interval $x > 0$,
$$y_2' = -x^{-2}$$
$$y_2'' = 2x^{-3}$$
so
$$x^2y_2'' - 2y_2 = x^2 2x^{-3} - 2^{-1} = 2x^{-1} - 2x^{-1} = 0.$$
Therefore both functions are solutions on the interval $x > 0$.

(b) The Wronskian is
$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = x^2(-x^{-2}) - x^{-1}(2x) = -1 - 2 = -3$$
which is never zero on $x > 0$, so the functions y_1 and y_2 are linearly independent on $x > 0$.

(c) By the previous results, the general solution to the differential equation is

$$y = c_1 x^2 + c_2 x^{-1}.$$

Differentiating,

$$y' = 2c_1 x - c_2 x^{-2}.$$

Setting $x = 1$,

$$y(1) = c_1 + c_2 = -2$$
$$y'(1) = 2c_1 - c_2 = -7.$$

Adding the above equations gives $c_1 = -3$. Multiplying the first by -2 and adding gives $c_2 = 1$. Therefore the solution to the initial value problem is

$$y = -3x^2 + x^{-1}.$$

Check!

3. (It may be wise to check that $y = x^{-1}$ is a solution before going too far into this.) We use variation of parameters. Suppose our second solution is of the form

$$y_2 = u(x)y_1$$

where $y_1 = x^{-1}$. Then

$$y_2' = u'y_1 + uy_1'$$
$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

and

$$x^2y_2'' - 2xy_2' - 4y_2 = x^2y_1u'' + x^2y_1'u' - 2xy_1u' + (x^2y_1'' - 2xy_1' - 4y_1)u = xu'' - 3u'$$

because y_1 is a solution of the differential equation. Therefore we must have

$$xu'' - 3u' = 0.$$

Letting $u' = v$ the equation becomes first order linear:

$$xv' - 3v = 0.$$

A solution can be found as in question 1; for example, $v = 4x^3$ is a solution. Then we have

$$u = \int v \, dx = \int 4x^3 \, dx = x^4,$$

so

$$y_2 = uy_1 = x^3$$

should be a second linearly independent solution to the original equation. Check. (You'll find that the answer is wrong; fix it!)

4. This is an Euler equation, which we did not study. You might try finding a solution of the form $y = x^m$, and then finding a second solution by variation of parameters. If that doesn't work, don't worry about it.

5. The auxiliary equation is

$$m^2 + 2m + 2 = 0$$

with roots $-1 \pm i$. Therefore the general solution is

$$y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t.$$

The initial conditions require that

$$\begin{aligned} c_1 &= 2 \\ -c_1 + c_2 &= 1 \end{aligned}$$

which implies that the solution to the initial value problem is

$$y = 2e^{-t} \cos t + 3e^{-t} \sin t.$$

Check!

6. The homogenous equation is $y'' + y = 0$ with fundamental system of solutions $\cos x$, $\sin x$. Now we form three determinants:

$$\begin{aligned} \Delta &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1, \\ \Delta_1 &= \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = \sin x \tan x, \\ \Delta &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \sin x. \end{aligned}$$

Taking quotients and integrating (by parts in the first case),

$$\begin{aligned} c_1(x) &= \int \frac{\Delta_1}{\Delta}\, dx = \int \sin x \tan x \, dx = -\tan x \cos x + \int \cos x \sec^2 x \; x = -\sin x + \ln|\tan x + \sec x| \\ c_2(x) &= \int \frac{\Delta_2}{\Delta}\, dx = \int \sin x \, dx = -\cos x. \end{aligned}$$

It follows that the general solution (on the interval I) is

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x + (-\sin x + \ln|\tan x + \sec x|) \cos x - \cos x \sin x \\ &= c_1 \cos x + c_2 \sin x + \cos x \ln|\tan x + \sec x|. \end{aligned}$$

Check!

7. (a) Just like a question in one of the problem sets.
   (b) This one is tricky. There is no formula for the Laplace transform of a product. You might try representing $\sin 2t \sin 5t$ as a convolution, but it's not at all clear to me how that might work. Instead, I suggest using the trig identities

$$\begin{aligned} \cos(2t - 7t) &= \cos 2t \cos 75 + \sin 2t \sin 7t \\ \cos(2t + 7t) &= \cos 2t \cos 7t - \sin 2t \sin 7t. \end{aligned}$$

   Subtracting the second from the first gives

$$\sin 2t \sin 7t = \frac{1}{2} \cos 5t - \frac{1}{2} \cos 9t.$$

   Taking the Laplace transform of the above gives

$$\mathscr{L}\{\sin 2t \sin 7t\} = \frac{1}{2} \frac{s}{s^2 + 25} - \frac{1}{2} \frac{s}{s^2 + 81}.$$

   Offhand, I can't think of a good way to check the result.

8. The usual, except substituting $s = 1$ and $s = 2$ doesn't get all the coefficients. Try substituting another value of s, or taking a derivative, or comparing coefficients of some power of s.

9. Taking the Laplace transform of the equation,

$$s^2Y(s) - sy(0) - y'(0) - 3sY(s) - 3y(0) + 2Y(s) = \frac{s}{s^2+1}.$$

Using the initial conditions and solving for $Y(s)$,

$$Y(s) = \frac{s}{(s^2+1)(s^2-3s+2)} + \frac{1}{s^2-3s+2}.$$

Now expanding in partial fractions,

$$\frac{1}{s^2-3s+2} = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}$$

and

$$\frac{s}{(s^2+1)(s^2-3s+2)} = \frac{s}{(s^2+1)(s-2)} - \frac{s}{(s^2+1)(s-1)} = \frac{-(5/2)s-(5/2}{s^2+1} + \frac{5/2}{s-2} - \frac{-(1/2)s+(1/2)}{s^2+1} - \frac{1/2}{s-1}.$$

(Check! There are other ways to obtain the partial fraction decomposition.) Assembling the above,

$$Y(s) = \frac{7}{2}\frac{1}{s-2} - \frac{3}{2}\frac{1}{s-1} - 2\frac{s}{s^2+1} - \frac{9}{2}\frac{1}{s^2+1}.$$

Taking the inverse Laplace transform,

$$y(t) = \frac{7}{2}e^{2t} - \frac{3}{2}e^{t} - 2\cos t - \frac{9}{2}\sin t.$$

Checking, you'll find that as usual I've made a mistake in the partial fractions decomposition which you should try to fix.

10. Let's solve this one by diagonalization (eigenvalues and eigenvectors). Call the matrix A. Then the characteristic polynomial is

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(\lambda-2)(\lambda-5).$$

So the eigenvalues are $\lambda = 1, 2, 5$. For the eigenvector with eigenvalue 1 we solve the system with augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right].$$

Adding -1 times row 3 to row 2, and then swapping rows around,

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Dividing row 1 by 2,

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so an eigenvector corresponding to $\lambda = 1$ is

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}.$$

You should check that eigenvector by doing the matrix multiplication. Similarly, for $\lambda = 2$, we have to solve the system

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right].$$

Adding 2 times row 1 to row 2, then multiplying row 1 by 1, then Multiplying row 3 by 1/2, then adding −1 times row 2 to row 3 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

with solution

$$\mathbf{v}_2 = \left[\begin{array}{c} 0 \\ 1 \\ -1 \end{array}\right].$$

You should check that $\mathbf{v}_2$ is an eigenvector for A with eigenvalue 2. Finally, to find an eigenvector for $\lambda = 5$, solve the system

$$\left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right].$$

Multiplying row 1 by −1/4, then adding −2 times row 1 to row 2, then adding row 2 to row 3 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

with a solution

$$\mathbf{v}_3 = \left[\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right].$$

Check that the above is an eigenvector. Assembling all the above information, a general solution for the system is

$$X = c_1 \left[\begin{array}{c} 2 \\ -3 \\ 2 \end{array}\right] e^t + c_2 \left[\begin{array}{c} 0 \\ 1 \\ -1 \end{array}\right] e^{2t} + c_3 \left[\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right] e^{5t}.$$

It might be a good idea to check the solution, although all the required checking has been done already.