Math 111 - 001 HW #4

8. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{x^a-1}{x^b-1} \overset{\text{H}}{=} \lim\limits_{x\to 1}\dfrac{ax^{a-1}}{bx^{b-1}} = \dfrac{a}{b}$

10. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\tan 5x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{4\cos 4x}{5\sec^2(5x)} = \dfrac{4(1)}{5(1)^2} = \dfrac{4}{5}$

12. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0}\dfrac{e^{3t}-1}{t} \overset{\text{H}}{=} \lim\limits_{t\to 0}\dfrac{3e^{3t}}{1} = 3$

16. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{x+x^2}{1-2x^2} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1+2x}{-4x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{2}{-4} = -\dfrac{1}{2}.$

A better method is to divide the numerator and the denominator by x^2: $\lim\limits_{x\to\infty}\dfrac{x+x^2}{1-2x^2} = \lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+1}{\frac{1}{x^2}-2} = \dfrac{0+1}{0-2} = -\dfrac{1}{2}.$

22. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{e^x-1-x-\frac{1}{2}x^2}{x^3} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{e^x-1-x}{3x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{e^x-1}{6x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{e^x}{6} = \dfrac{1}{6}$

32. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{x}{\tan^{-1}(4x)} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{1}{\dfrac{1}{1+(4x)^2}\cdot 4} = \lim\limits_{x\to 0}\dfrac{1+16x^2}{4} = \dfrac{1}{4}$

36. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-2}{1-\cos x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{e^x+e^{-x}}{\cos x} = \dfrac{1+1}{1} = 2$

50. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{x(-\sin x)+\cos x - \cos x}{x\cos x + \sin x}$$

$$= -\lim_{x\to 0}\frac{x\sin x}{x\cos x + \sin x} \overset{\text{H}}{=} -\lim_{x\to 0}\frac{x\cos x + \sin x}{x(-\sin x)+\cos x+\cos x} = -\frac{0+0}{0+1+1} = 0$$

54. $y = (\tan 2x)^x \quad\Rightarrow\quad \ln y = x\cdot\ln\tan 2x$, so

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\cdot\ln\tan 2x = \lim_{x\to 0^+}\frac{\ln\tan 2x}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2} = \lim_{x\to 0^+}\frac{-2x^2\cos 2x}{\sin 2x\,\cos^2 2x}$$

$$= \lim_{x\to 0^+}\frac{2x}{\sin 2x}\cdot\lim_{x\to 0^+}\frac{-x}{\cos 2x} = 1\cdot 0 = 0 \quad\Rightarrow$$

$$\lim_{x\to 0^+}(\tan 2x)^x = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

56. $y = \left(1+\dfrac{a}{x}\right)^{bx} \quad\Rightarrow\quad \ln y = bx\ln\left(1+\dfrac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{ab}{1+a/x} = ab \quad\Rightarrow$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

60. $y = (e^x+x)^{1/x} \quad\Rightarrow\quad \ln y = \dfrac{1}{x}\ln(e^x+x)$,

$$\text{so } \lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(e^x+x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x+1}{e^x+x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{e^x+1} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{e^x} = 1 \quad\Rightarrow$$

$$\lim_{x\to\infty}(e^x+x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e.$$

64. $y = \left(\dfrac{2x-3}{2x+5}\right)^{2x+1} \quad\Rightarrow\quad \ln y = (2x+1)\ln\left(\dfrac{2x-3}{2x+5}\right) \quad\Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2} = \lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8 \quad\Rightarrow\quad \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = e^{-8}$$