Complete convergence for arrays of negatively dependent random variables

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Abstract

We present a general result establishing complete convergence for the row sums of an array of rowwise negatively dependent random variables. From this result, we obtain many complete convergence results for weighted sums of negatively dependent random variables.

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1 Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins (1947) as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

In view of the Borel-Cantelli lemma, this implies that $U_n \to \theta$ almost surely. The converse is true if $\{U_n, n \geq 1\}$ are independent random variables. Hsu and Robbins (1947) and Katz (1963) ($p = 1$ and $1 < p < 2$, respectively) proved that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with mean zero and $E|X_1|^{2p} < \infty$, then $\sum_{i=1}^{n} X_i/n^{1/p}$ converges completely to zero.

The paper Hsu and Robbins (1947) initiated numerous explorations of the complete convergence of sums of independent random variables. Their research was continued by Erdös (1949, 1950), Spitzer (1956), Baum and Katz (1965), and Gut (1992). This subject is actively discussed in scientific press during the last few decades. For example, Hu *et al.* (1989) extended the result of Hsu-Robbins-Katz to the case where $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of rowwise independent random variables which are stochastically dominated by a random variable X satisfying $E|X|^{2p} < \infty$ for some $1 \leq p < 2$.

The papers Kruglov *et al.* (2006) and Sung (2007) contain, up to our knowledge, the most general theorems that provide sufficient conditions for complete convergence for sums of arrays of rowwise independent random variables.

In the following, let $\{k_n, n \geq 1\}$ be a sequence of positive integers. In general the case $k_n = \infty$ is not precluded. When $k_n = \infty$, we will assume that $\sum_{i=1}^{\infty} X_{ni}$ converges almost surely. Recall that an array $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ of random variables is said to be *stochastically dominated* by a random variable X if there exists a positive constant $C > 0$ such that

$$P\{|X_{ni}| > x\} \leq CP\{|X| > x\} \text{ for all } x > 0, 1 \leq i \leq k_n, \text{ and } n \geq 1.$$

Recently, some complete convergence theorems for negatively dependent random variables have been obtained by many authors (see, for example, Giuliano *et al.* (2008) and Taylor *et al.* (2002) and references in these papers). Taylor et al. (2002) extended the result of Hu et al. (1989) to the array of rowwise negatively dependent random variables. Giuliano *et al.* (2008) considered so-called acceptable random variable, which is more general notion than negatively dependency.

The finite set of random variables $X_1, \cdots, X_n$ is said to be *negatively dependent* if

$$P\{X_1 \leq x_1, \cdots, X_n \leq x_n\} \leq P\{X_1 \leq x_1\} \cdots P\{X_n \leq x_n\}$$

and

$$P\{X_1 > x_1, \cdots, X_n > x_n\} \le P\{X_1 > x_1\} \cdots P\{X_n > x_n\}$$

for all real numbers $x_1, \cdots, x_n$. An infinite sequence $\{X_n, n \ge 1\}$ is said to be negatively dependent if every finite subset of the sequence $\{X_1, \cdots, X_n\}$ is negatively dependent.

In this paper, we present a general result establishing complete convergence for the row sums of an array of rowwise negatively dependent random variables. It also specifies the corresponding rate of convergence. From this result, we obtain many complete convergence results for negatively dependent random variables. As a corollary, the result of Taylor *et al.* (2002) is obtained.

Throughout this paper, C denotes a positive constant which may be different in various places, and it is convenient to define $\log x = \max\{1, \ln x\}$, where $\ln x$ denotes the natural logarithm.

2 Preliminary lemmas

To prove our main result, the following lemmas are needed. The first two lemmas are well known and can be found, for example, in Taylor *et al.* (2002).

Lemma 1. *Let $\{X_n, n \ge 1\}$ be a sequence of negatively dependent random variables and $\{f_n, n \ge 1\}$ be a sequence of Borel functions all of which are monotone increasing (or monotone decreasing), then $\{f_n(X_n), n \ge 1\}$ is a sequence of negatively dependent random variables.*

The second lemma mainly states that negatively dependent random variables are negatively correlated.

Lemma 2. *Let $X_1, \cdots, X_n$ be negatively dependent integrable random variables. Then*

$$E \prod_{i=1}^{n} X_i \le \prod_{i=1}^{n} EX_i.$$

The following lemma plays an essential role in our main result. Of course, this lemma is of interest only if positive constants d_i, and hence second moments $EX_i^2, 1 \le i \le n$, are close to zero (at least less than one). Otherwise we have alternative so-called subgaussian estimations, see, for example, Giuliano *et al.* (2008).

Lemma 3. *Let $X_1, \cdots, X_n$ be negatively dependent mean zero random variables such that*

$$|X_i| \le d_i, \quad 1 \le i \le n$$

for a sequence of positive constants $d_1, \cdots, d_n$. Then for any $t > 0$,

$$E \exp\left\{ t \sum_{i=1}^{n} X_i \right\} \le \exp\left\{ \frac{t^2}{2} \sum_{i=1}^{n} e^{td_i} E X_i^2 \right\}.$$

Proof. From the inequality $e^x \le 1 + x + \frac{x^2}{2} e^{|x|}$, which is true for all x, we have

$$\begin{aligned} Ee^{tX_i} &\le 1 + tEX_i + \frac{t^2}{2} E\left(X_i^2 e^{t|X_i|}\right) \\ &= 1 + \frac{t^2}{2} E\left(X_i^2 e^{t|X_i|}\right) \text{ (since the } X_i \text{ have mean zero)} \\ &\le 1 + \frac{t^2}{2} e^{td_i} E X_i^2 \le \exp\left\{ \frac{t^2}{2} e^{td_i} E X_i^2 \right\}, \end{aligned}$$

since $1 + x \le e^x$ for all x. It follows by Lemmas 1 and 2 that

$$E \exp\left\{ t \sum_{i=1}^{n} X_i \right\} \le \prod_{i=1}^{n} E e^{tX_i} \le \prod_{i=1}^{n} \exp\left\{ \frac{t^2}{2} e^{td_i} E X_i^2 \right\}$$

$$= \exp\left\{ \frac{t^2}{2} \sum_{i=1}^{n} e^{td_i} E X_i^2 \right\}. \square$$

3 Main result

With the preliminary lemmas, we now state and prove our main result.

Theorem. *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise negatively dependent random variables, $\{a_n, n \ge 1\}$ be a sequence of positive constants, and $\{b_n, n \ge 1\}$ be a sequence of positive constants such that $\lim_{n\to\infty} b_n = \infty$. Suppose that*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P\{|X_{ni}| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$,

(ii) $\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} P\{|X_{ni}| > 1/b_n\} \right)^{N_1} < \infty$ *for some* $N_1 > 0$,

(iii) $b_n \sum_{i=1}^{k_n} EX_{ni}^2 I\{|X_{ni}| \le 1/b_n\} \to 0$ *as* $n \to \infty$, *and*

(iv) $\sum_{n=1}^{\infty} a_n \exp\{-N_2 b_n\} < \infty$ *for some* $N_2 > 0$.

Then $\sum_{n=1}^{\infty} a_n P\left\{ \left| \sum_{i=1}^{k_n} X_{ni} - EX_{ni} I\{|X_{ni}| \le 1/b_n\} \right| > \varepsilon \right\} < \infty$ *for all* $\varepsilon > 0$.

Proof. The set of all natural numbers is partitioned into two subsets

$$A' = \left\{ n : \sum_{i=1}^{k_n} P\{|X_{ni}| > 1/b_n\} \le 1 \right\}$$

and

$$A'' = \left\{ n : \sum_{i=1}^{k_n} P\{|X_{ni}| > 1/b_n\} > 1 \right\}.$$

Applying (ii), we obtain

$$\sum_{n \in A''} a_n P\left\{ \left| \sum_{i=1}^{k_n} X_{ni} - EX_{ni} I\{|X_{ni}| \le 1/b_n\} \right| > \varepsilon \right\}$$

$$\le \sum_{n \in A''} a_n \le \sum_{n \in A''} a_n \left(\sum_{i=1}^{k_n} P\{|X_{ni}| > 1/b_n\} \right)^{N_1} < \infty.$$

Hence it is enough to show that

$$\sum_{n \in A'} a_n P\left\{ \left| \sum_{i=1}^{k_n} X_{ni} - EX_{ni} I\{|X_{ni}| \le 1/b_n\} \right| > \varepsilon \right\} < \infty \text{ for all } \varepsilon > 0.$$

For $1 \le i \le k_n$ and $n \ge 1$ define

$$\begin{aligned}
Y_{ni} &= X_{ni} I\{|X_{ni}| \le 1/b_n\} + \frac{1}{b_n} I\{X_{ni} > 1/b_n\} - \frac{1}{b_n} I\{X_{ni} < -1/b_n\}, \\
U_{ni} &= \frac{1}{b_n} \left(I\{X_{ni} < -1/b_n\} - P\{X_{ni} < -1/b_n\} \right), \\
V_{ni} &= -\frac{1}{b_n} \left(I\{X_{ni} > 1/b_n\} - P\{X_{ni} > 1/b_n\} \right), \\
Z_{ni} &= X_{ni} I\{1/b_n < |X_{ni}| \le \varepsilon/(4[N_1 + 1])\}.
\end{aligned}$$

Then $\{Y_{ni} - EY_{ni}, 1 \le i \le k_n, n \ge 1\}$, $\{U_{ni}, 1 \le i \le k_n, n \ge 1\}$ and $\{V_{ni}, 1 \le i \le k_n, n \ge 1\}$ are arrays of rowwise negatively dependent random variables by Lemma 1.

Note that if we define

$$W_{ni} = \frac{1}{b_n}\left(I\{|X_{ni}| > 1/b_n\} - P\{|X_{ni}| > 1/b_n\}\right),$$

then we cannot state that $\{W_{ni}, 1 \le i \le k_n, n \ge 1\}$ is an array of negatively dependent random variables. This is a sort of the main disadvantage when we are dealing with negatively dependent random variables.

Since $\lim_{n\to\infty} b_n = \infty$, there exists a positive integer M such that

$$\frac{\varepsilon}{4[N_1+1]} > \frac{1}{b_n}$$

for all $n > M$. For $n > M$, we can write that

$$\sum_{i=1}^{k_n} X_{ni} - EX_{ni}I\{|X_{ni}| \le 1/b_n\}$$

$$= \sum_{i=1}^{k_n}(Y_{ni} - EY_{ni}) + \sum_{i=1}^{k_n} U_{ni} + \sum_{i=1}^{k_n} V_{ni} + \sum_{i=1}^{k_n} Z_{ni} + \sum_{i=1}^{k_n} X_{ni}I\{|X_{ni}| > \varepsilon/(4[N_1+1])\}.$$

It follows that

$$\begin{aligned}
&\sum_{n>M,\ n\in A'} a_n P\left\{\sum_{i=1}^{k_n} X_{ni} - EX_{ni}I\{|X_{ni}| \le 1/b_n\} > \varepsilon\right\} \\
\le\ & \sum_{n>M,\ n\in A'} a_n P\left\{\sum_{i=1}^{k_n} Y_{ni} - EY_{ni} > \varepsilon/4\right\} \\
&+ \sum_{n>M,\ n\in A'} a_n P\left\{\sum_{i=1}^{k_n} U_{ni} > \varepsilon/4\right\} \\
&+ \sum_{n>M,\ n\in A'} a_n P\left\{\sum_{i=1}^{k_n} V_{ni} > \varepsilon/4\right\} \\
&+ \sum_{n>M,\ n\in A'} a_n P\left\{\sum_{i=1}^{k_n} Z_{ni} > \varepsilon/4\right\} \\
&+ \sum_{n>M,\ n\in A'} a_n P\left\{|X_{ni}| > \varepsilon/(4[N_1+1]) \text{ for some } 1 \le i \le k_n\right\} \\
=\ & : I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now we estimate each sum separately.

For I_1, we note that $|Y_{ni}| \le 1/b_n$ and

$$Y_{ni}^2 = X_{ni}^2 I\{|X_{ni}| \le 1/b_n\} + (1/b_n)^2 I\{|X_{ni}| > 1/b_n\}.$$

Moreover, we have that

$$(1/b_n)\sum_{i=1}^{k_n} P\{|X_{ni}| > 1/b_n\} = o(1) \text{ for } n \in A'.$$

By Lemma 3 with $t = 4(N_2+1)b_n/\varepsilon$, we obtain that for $n \in A'$

$$\begin{aligned}
& P\left\{\sum_{i=1}^{k_n}(Y_{ni} - EY_{ni}) > \varepsilon/4\right\} \\
\le \quad & \exp\left\{-\frac{t\varepsilon}{4}\right\} E\exp\left\{t\sum_{i=1}^{k_n} Y_{ni} - EY_{ni}\right\} \\
\le \quad & \exp\left\{-\frac{t\varepsilon}{4}\right\} \exp\left\{\frac{t^2}{2}e^{2t/b_n}\sum_{i=1}^{k_n} E(Y_{ni} - EY_{ni})^2\right\} \\
\le \quad & \exp\left\{-\frac{t\varepsilon}{4}\right\} \exp\left\{\frac{t^2}{2}e^{2t/b_n}\sum_{i=1}^{k_n} EY_{ni}^2\right\} \\
= \quad & \exp\left\{-\frac{t\varepsilon}{4}\right\} \exp\left\{\frac{t^2}{2}e^{2t/b_n}\sum_{i=1}^{k_n} EX_{ni}^2 I\{|X_{ni}| \le 1/b_n\} + \frac{1}{b_n^2}P\{|X_{ni}| > 1/b_n\}\right\} \\
\le \quad & \exp\left\{-(N_2+1)b_n + 8(N_2+1)^2 e^{8(N_2+1)/\varepsilon}\varepsilon^{-2}o(1)b_n\right\} \text{ by (iii)} \\
= \quad & \exp\left\{-\left(N_2+1-o(1)\right)b_n\right\} \\
\le \quad & \exp\{-N_2 b_n\}
\end{aligned}$$

for all large n. Thus $I_1 < \infty$ by (iv).

For I_2, we observe that $|U_{ni}| \le 1/b_n$ and $EU_{ni}^2 \le P(|X_{ni}| > 1/b_n)/b_n^2$. Hence

$$\sum_{i=1}^{k_n} EU_{ni}^2 \le \frac{1}{b_n^2}\sum_{i=1}^{k_n} P\{|X_{ni}| > 1/b_n\} = \frac{1}{b_n}o(1) \text{ for } n \in A'.$$

By Lemma 3 with $t = 4(N_2+1)b_n/\varepsilon$, we obtain that for $n \in A'$,

$$
\begin{aligned}
& P\{\sum_{i=1}^{k_n} U_{ni} > \varepsilon/4\} \leq \exp\{-t\varepsilon/4\} E \exp\left\{ t \sum_{i=1}^{k_n} U_{ni} \right\} \\
\leq \;& \exp(-t\varepsilon/4) \exp\left\{ \frac{t^2}{2} e^{t/b_n} \sum_{i=1}^{k_n} EU_{ni}^2 \right\} \\
\leq \;& \exp\left\{ -(N_2+1)b_n + 8(N_2+1)^2 e^{4(N_2+1)/\varepsilon} \varepsilon^{-2} o(1) b_n \right\} \\
\leq \;& \exp\{-N_2 b_n\}
\end{aligned}
$$

for all large n. Thus $I_2 < \infty$ by (iv).

Similarly to I_2, we get $I_3 < \infty$.

For I_4, we note that

$$
\begin{aligned}
& P\left\{ \sum_{i=1}^{k_n} Z_{ni} > \varepsilon/4 \right\} \leq P\{ \text{ at least } [N_1+1] \text{ of } Z_{ni} \neq 0 \} \\
& \quad \text{because } Z_{ni} < \varepsilon/4[N_1+1] \\
= \;& P\{ \text{ at least } [N_1+1] \text{ of } X_{ni} \text{ have the property } 1/b_n < |X_{ni}| \leq \varepsilon/(4[N_1+1]) \} \\
\leq \;& \sum_{j_1 < \cdots < j_{[N_1+1]}} P\{X_{n,j_1} > 1/b_n, \cdots, X_{n,j_{[N_1+1]}} > 1/b_n\} \\
& \text{where the summation is taken for all } [N_1+1]-\text{tuple } (j_1, \cdots, j_{[N_1+1]}) \\
& \quad \text{with } j_1 < \cdots < j_{[N_1+1]} \text{ and } j_i = 1, \cdots, k_n \text{ for each } i \\
\leq \;& \sum_{j_1 < \cdots < j_{[N_1+1]}} P\{X_{n,j_1} > 1/b_n\} \cdots P\{X_{n,j_{[N_1+1]}} > 1/b_n\} \text{ by negative dependence} \\
= \;& \sum_{j_1 < \cdots < j_{[N_1+1]}} \prod_{k=1}^{[N_1+1]} P\{X_{n,j_k} > 1/b_n\} \\
\leq \;& \sum_{j_1, \cdots, j_{[N_1+1]}} \prod_{i=1}^{[N_1+1]} P\{X_{n,j_i} > 1/b_n\} \\
& \text{where the summation is taken for all possible } [N_1+1]-\text{tuple } (j_1, \cdots, j_{[N_1+1]}) \\
& \quad \text{and } j_i = 1, \cdots, k_n \text{ for each } i \\
= \;& \left(\sum_{i=1}^{k_n} P\{|X_{ni}| > 1/b_n\} \right)^{[N_1+1]},
\end{aligned}
$$

Thus $I_4 < \infty$ by (ii).

Obviously $I_5 < \infty$ by (i).

Therefore we have that

$$\sum_{n>M,n\in A'} a_n P(\sum_{i=1}^{k_n}(X_{ni} - EX_{ni}I(|X_{ni}| \leq 1/b_n)) > \varepsilon) < \infty.$$

Since $\{-X_{ni}\}$ is also an array of rowwise negatively dependent random variables, we can replace X_{ni} by $-X_{ni}$ in the above statement. That is,

$$\sum_{n>M,n\in A'} a_n P(\sum_{i=1}^{k_n}(X_{ni} - EX_{ni}I(|X_{ni}| \leq 1/b_n)) < -\varepsilon) < \infty.$$

□

Remark 1. In view of assumption (iii), it is interesting to consider sequences $\{b_n, n \geq 1\}$ that increase to infinity as slow as possible for (iv) still be true. If the sequence $\{a_n, n \geq 1\}$ has a polynomial growth or a constant (that is, $a_n = n^t, t \geq 0$), then the good choice is $b_n = \log n, n \geq 1$, which has been explored in Sung (2007) for the case of rowwise independent arrays. But our Theorem can be applied for sequences $\{a_n, n \geq 1\}$ with a different than polynomial behaviour. The main idea is that we can link sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ according to assumption (iv).

4 Corollaries

Theorem presented and proved in the previous section can be applied in different situations for various choices of weights and moment conditions.

Corollary 1. *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise negatively dependent mean zero random variables which are stochastically dominated by a random variable X with $E|X|^{2p} < \infty$ for some $p \geq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers and $\{b_n, n \geq 1\}$ be a sequence of positive constants such that*

(a) $\lim_{n\to\infty} b_n = \infty$,

(b) $b_n = O(n^q)$ *for some* $0 < q < 1/(2p)$,

(c) $\sum_{n=1}^{\infty} \exp\{-N_2 b_n\} < \infty$ *for some* $N_2 > 0$.

(d) $b_n \sum_{i=1}^{n} a_{ni}^2 = o(1)$ *as* $n \to \infty$,

(e) $\max_{1 \le i \le n} |a_{ni}| = O(1/n^{1/p})$.

Then $\sum_{i=1}^n a_{ni} X_{ni} \to 0$ *completely.*

Proof. Without loss of generality, we may assume that $a_{ni} \geq 0$ for $1 \leq i \leq n$ and $n \geq 1$. Otherwise we prove the result separately for two arrays of constants $\{a_{ni}^+, 1 \leq i \leq n, n \geq 1\}$ and $\{a_{ni}^-, 1 \leq i \leq n, n \geq 1\}$, where we use the notations $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$. Then $\{a_{ni} X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of rowwise negatively dependent random variables by Lemma 1. We may also assume that $\max_{1 \leq i \leq n} a_{ni} \leq 1/n^{1/p}$.

We will apply Theorem with $a_n = 1, n \geq 1$ and X_{ni} replaced by $a_{ni} X_{ni}, 1 \leq i \leq n, n \geq 1$.

In order to check condition (i) of Theorem, note that by the stochastic domination hypothesis

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{i=1}^{n} P\{|a_{ni} X_{ni}| > \varepsilon\} &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} P\{|X_{ni}| > \varepsilon n^{1/p}\} \\ &\leq C \sum_{n=1}^{\infty} n P\{|X| > \varepsilon n^{1/p}\}. \end{aligned}$$

The sum $\sum_{n=1}^{\infty} n P\{|X|^p > n\} < \infty$ if and only if $E|X|^{2p} < \infty$. Thus condition (i) of Theorem holds.

For condition (ii), taking $N_1 > 1/(1-2pq)$, we have by Markov's inequality and the stochastic domination hypothesis that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} P\{|a_{ni} X_{ni}| > 1/b_n\} \right)^{N_1} &\leq \sum_{n=1}^{\infty} \left(b_n^{2p} \sum_{i=1}^{n} |a_{ni}|^{2p} E|X_{ni}|^{2p} \right)^{N_1} \\ \leq & \sum_{n=1}^{\infty} \left(CE|X|^{2p} b_n^{2p}/n \right)^{N_1} \text{ by assumption (e)} \\ < & \ \infty \text{ by assumption (b) and the fact that } N_1 > 1/(1-2pq). \end{aligned}$$

Thus condition (ii) holds.

For condition (iii),

$$b_n \sum_{i=1}^{n} E(a_{ni}X_{ni})^2 I(|a_{ni}X_{ni}| \le 1/b_n) \le b_n \sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2$$
$$\le \ CEX^2 b_n \sum_{i=1}^{n} a_{ni}^2 \to 0 \text{ by (d)}$$

Thus condition (iii) holds.

Condition (iv) holds by the assumption (c).

By Theorem we obtain that

$$\sum_{n=1}^{\infty} P\left\{ |\sum_{i=1}^{n} a_{ni}(X_{ni} - EX_{ni}I\{|a_{ni}X_{ni}| \le 1/b_n\})| > \varepsilon \right\} < \infty \text{ for all } \varepsilon > 0.$$

It remains to show that

$$\sum_{i=1}^{n} a_{ni}EX_{ni}I\{|a_{ni}X_{ni}| \le 1/b_n\} \to 0.$$

Since $EX_{ni} = 0$, $EX_{ni}I\{|a_{ni}X_{ni}| \le 1/b_n\} = -EX_{ni}I\{|a_{ni}X_{ni}| > 1/b_n\}$. It follows that

$$|\sum_{i=1}^{n} a_{ni}EX_{ni}I\{|a_{ni}X_{ni}| \le 1/b_n\}| \le \sum_{i=1}^{n} |a_{ni}|E|X_{ni}|I\{|a_{ni}X_{ni}| > 1/b_n\}$$
$$\le \ \frac{1}{n^{1/p}} \sum_{i=1}^{n} E|X_{ni}|I\{|X_{ni}| > n^{1/p}/b_n\} \text{ by assumption (e)}$$
$$\le \ Cn^{1-\ 1/p}E|X|I\{|X| > n^{1/p}/b_n\}$$
$$\le \ Cn^{1-\ 1/p}E|X|^{2p}|X|^{1-2p}I\{|X| > n^{1/p}/b_n\}$$
$$\le \ CE|X|^{2p}n^{1-\ 1/p}\left(\frac{b_n}{n^{1/p}}\right)^{2p-1}$$
$$\le \ Cn^{-1/(2p)} \to 0$$

since $b_n < Cn^{1/(2p)}$ for n large enough. Thus the proof is completed. □

As a special case of Corollary 1, we get the following corollary which was proved by Taylor et al. (2002).

Corollary 2. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise negatively dependent mean zero random variables which are stochastically dominated by a random variable X with $E|X|^{2p} < \infty$ for some $1 \leq p < 2$. Then

$$\sum_{i=1}^{n} X_{ni}/n^{1/p} \to 0 \text{ completely.}$$

Proof. Let $a_{ni} = 1/n^{1/p}$ for $1 \leq i \leq n$ and $n \geq 1$. Then conditions of Corollary 1 are trivially satisfied with $b_n = n^q$ for some $0 < q < \min\left\{\frac{1}{2p}, \frac{2}{p} - 1\right\}$.□

Corollary 3. *Let $t > -1, p > 0$, and $\beta \in \mathbf{R}$. Denote $\Delta = p(t + \beta + 1)$ and assume that $\Delta \geq 1$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise negatively dependent mean zero random variables which are stochastically dominated by a random variable X with $E|X|^{\Delta} < \infty$. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be a bounded array of real numbers such that*

(1) $\sum_{i=1}^{\infty} |a_{ni}|^q = O(n^\beta)$ *for some $q < \Delta$,*

(2) *If $\Delta \geq 2$, then $\sum_{i=1}^{\infty} a_{ni}^2 = O(n^\gamma)$ for some $\gamma < 2/p$.*

Then

$$\sum_{n=1}^{\infty} n^t P\left\{\left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| / n^{1/p} > \varepsilon\right\} < \infty \text{ for all } \varepsilon > 0.$$

Proof. The same as in the proof of Corollary 1, without loss of generality, we may assume that $a_{ni} \geq 0$ for $i \geq 1, n \geq 1$. Then $\{a_{ni}X_{ni}/n^{1/p}, i \geq 1, n \geq 1\}$ is an array of rowwise negatively dependent random variables by Lemma 1. We will apply Theorem with $a_n = n^t, n \geq 1$ and X_{ni} replaced by $a_{ni}X_{ni}/n^{1/p}, i \geq 1, n \geq 1$.

Consider the sequence $b_n = n^\alpha, n \geq 1$, where $0 < \alpha < \frac{t+1}{\Delta}$. For the case $\Delta \geq 2$ we require additionally that $0 < \alpha < \frac{2}{p} - \gamma$.

The fact that

$$\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} P(|a_{ni} n^{-1/p} X_{ni}| > \varepsilon) \leq CE|X|^{p(t+\beta+1)} < \infty$$

was established in many papers, see for example, Hu *et al.* (2002) (beginning of the proof of Theorem 3.1), Ahmed *et al.* (2002) (beginning of the proof of Theorem 3.1) and Sung (2007) (beginning of the proof of Theorem 2 and Lemma 3). We also note that the proof presented in Hu *et al.* (2002) is rather complicated once it uses the Stieltjes integration technique, summation by parts lemma and so on. The proof presented in Ahmed *et al.* (2002) is much more elegant. Also, Hu *et al.* (2002) and Ahmed *et al.* (2002) are dealing with an array of constants $\{a_{ni}X_{ni}, i \geq 1, n \geq 1\}$ rather than the array $\{a_{ni}X_{ni}/n^{1/p}, i \geq 1, n \geq 1\}$ which is considered in Sung (2007) and this paper.

According to the inequality presented above, condition (i) of Theorem holds.

For (ii), taking $N_1 > \frac{t+1}{t+1-\alpha\Delta} > 0$, we have by Markov's inequality, $|a_{ni}| = O(1)$, and (1) that

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^t \left(\sum_{i=1}^{\infty} P\{|a_{ni}n^{-1/p}X_{ni}| > 1/b_n\} \right)^{N_1} \\
\leq & \sum_{n=1}^{\infty} n^t \left(b_n^{\Delta} n^{-(t+\beta+1)} \sum_{i=1}^{\infty} |a_{ni}|^{\Delta} E|X_{ni}|^{\Delta} \right)^{N_1} \\
\leq & \; C \sum_{n=1}^{\infty} n^t \left(b_n^{\Delta} n^{-(t+\beta+1)} \sum_{i=1}^{\infty} |a_{ni}|^{q} |a_{ni}|^{\Delta-q} \right)^{N_1} \\
\leq & \; C \sum_{n=1}^{\infty} n^{t+\alpha\Delta N_1-(t+1)N_1} < \infty,
\end{aligned}$$

since $t + \alpha\Delta N_1 - (t+1)N_1 < -1$. Thus condition (ii) of Theorem holds.

For condition (iii) we consider two cases. If $1 \leq \Delta < 2$, by (1) we obtain

$$\begin{aligned}
& b_n \sum_{i=1}^{\infty} E(a_{ni}n^{-1/p}X_{ni})^2 I\{|a_{ni}n^{-1/p}X_{ni}| \leq 1/b_n\} \\
= & \; b_n \sum_{i=1}^{\infty} E|a_{ni}n^{-1/p}X_{ni}|^{\Delta} |a_{ni}n^{-1/p}X_{ni}|^{2-\Delta} I\{|a_{ni}n^{-1/p}X_{ni}| \leq 1/b_n\} \\
\leq & \; b_n^{\Delta-1} \sum_{i=1}^{\infty} E|a_{ni}n^{-1/p}X_{ni}|^{\Delta} I\{|a_{ni}n^{-1/p}X_{ni}| \leq 1/b_n\} \\
\leq & \; b_n^{\Delta-1} \sum_{i=1}^{\infty} E|a_{ni}n^{-1/p}X_{ni}|^{\Delta} \\
\leq & \; C b_n^{\Delta-1} E|X|^{\Delta} \sum_{i=1}^{\infty} |a_{ni}n^{-1/p}|^{\Delta}
\end{aligned}$$

$$
\begin{aligned}
&\leq \; Cn^{\alpha\Delta-\alpha-t-1} \\
&< \; Cn^{-\alpha} \to 0 \text{ as } n \to \infty
\end{aligned}
$$

by the choice of α.

If $\Delta \geq 2$, then by (2)

$$
\begin{aligned}
& b_n \sum_{i=1}^{\infty} E(a_{ni} n^{-1/p} X_{ni})^2 I\{|a_{ni} n^{-1/p} X_{ni}| \leq 1/b_n\} \\
\leq \; & C b_n E X^2 \sum_{i=1}^{\infty} a_{ni}^2 / n^{2/p} \\
\leq \; & C E X^2 n^{\alpha+\gamma-\ 2/p} \to 0 \text{ as } n \to \infty
\end{aligned}
$$

by the choice of α. Thus condition (iii) of Theorem holds.

Condition (iv) holds trivially.

Hence we get by Theorem that

$$
\sum_{n=1}^{\infty} n^t P \left\{ |\sum_{i=1}^{\infty} a_{ni} n^{-1/p} (X_{ni} - E X_{ni} I \left\{ |a_{ni} X_{ni}| \leq n^{1/p}/b_n \right\})| > \varepsilon \right\} < \infty \text{ for all } \varepsilon > 0.
$$

It remains to show that

$$
\sum_{i=1}^{\infty} a_{ni} n^{-1/p} E X_{ni} I\{|a_{ni} X_{ni}| \leq n^{1/p}/b_n\} \to 0.
$$

Since $EX_{ni} = 0$, $EX_{ni} I(|a_{ni} X_{ni}| \leq n^{1/p}/b_n) = -EX_{ni} I\{|a_{ni} X_{ni}| > n^{1/p}/b_n\}$.
It follows that

$$
\begin{aligned}
& \left| \sum_{i=1}^{\infty} a_{ni} n^{-1/p} E X_{ni} I\{|a_{ni} X_{ni}| \leq n^{1/p}/b_n\} \right| \\
\leq \; & n^{-1/p} \sum_{i=1}^{\infty} E|a_{ni} X_{ni}| I\{|a_{ni} X_{ni}| > n^{1/p}/b_n\} \\
\leq \; & n^{-1/p} (b_n/n^{1/p})^{\Delta-1} \sum_{i=1}^{\infty} E|a_{ni} X_{ni}|^{\Delta} I\{|a_{ni} X_{ni}| > n^{1/p}/b_n\} \\
\leq \; & C(b_n)^{\Delta-1} E|X|^{\Delta}/n^{t+1} \\
\leq \; & C n^{\alpha(\Delta-1)-t-1} \to 0,
\end{aligned}
$$

By the choice of α.

Thus the proof is completed. □

Remark 2. If $t < -1$, then the conclusion of Corollary 3 holds trivially. When $t \geq -1$, Sung (2007) proved Corollary 3 under the stronger condition that $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise independent random variables. However, the relatively important case $t = -1$ in Corollary 3 cannot be proved by using Theorem. We left as an open problem whether Corollary 3 holds for $t = -1$.

As a special case of Corollary 3, we get the following corollary.

Corollary 4. *Let $t > -1$ and $1 \leq p < 2$. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise negatively dependent mean zero random variables which are stochastically dominated by a random variable X with $E|X|^{p(t+2)} < \infty$. Then*

$$\sum_{n=1}^{\infty} n^t P\left\{\left|\sum_{i=1}^{n} X_{ni}\right| / n^{1/p} > \varepsilon\right\} < \infty \text{ for all } \varepsilon > 0.$$

Proof. Let $a_{ni} = 1$ for $1 \leq i \leq n$ and $a_{ni} = 0$ for $i > n$. Then, for $q < p(t+2)$, $\sum_{i=1}^{\infty} |a_{ni}|^q = n$. Thus, assumption (1) of Corollary 3 holds for $\beta = 1$. Since $1 \leq p < 2$, assumption (2) holds for $\gamma = 1$. Thus the result follows from Corollary 3. □

Remark 3. When $t = 0$, Corollary 4 is same as Corollary 2.

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