

Marcinkiewicz-Zygmund Type Law of Large Numbers for Double Arrays of Random Elements in Banach Spaces

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Abstract—In this paper we establish Marcinkiewicz-Zygmund type laws of large numbers for double arrays of random elements in Banach spaces. Our results extend those of Hong and Volodin [6].

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1. INTRODUCTION

Marcinkiewicz-Zygmund type strong laws of large numbers were studied by many authors. In 1981, Etemadi [3] proved that if $\{X_n; n \geq 1\}$ is a sequence of pairwise i.i.d. random variables with $EX_1 < \infty$, then $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (X_i - EX_1) = 0$ a.s.

Later, in 1985, Choi and Sung [2] have shown that if $\{X_n; n \geq 1\}$ are pairwise independent and are dominated in distribution by a random variable X with $E|X|^p(\log^+ |X|)^2 < \infty$, $1 < p < 2$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^n (X_i - EX_i) = 0 \text{ a.s.}$$

Recently, Hong and Hwang [5], Hong and Volodin [6] studied Marcinkiewicz-Zygmund strong law of large numbers for double sequence of random variables, Quang and Thanh [12] established the Marcinkiewicz-Zygmund strong law of large numbers for blockwise adapted sequence. In this paper, we extend the results of Hong and Volodin [6] to some special class of Banach spaces, so-called Banach spaces that satisfy the maximal Marcinkiewicz-Zygmund inequality with exponent p (see the definition below). This class includes Rademacher type p and martingale type p Banach spaces, $0 < p \leq 2$.

For $a, b \in \mathbb{R}$, $\max\{a, b\}$ will be denoted by $a \vee b$. Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

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2. PRELIMINARIES

Technical definitions relevant to the current work will be discussed in this section.

The Banach space \mathcal{X} is said to be of *Rademacher type* p ($1 \leq p \leq 2$) if there exists a constant $C < \infty$ such that

$$E \left\| \sum_{j=1}^n V_j \right\|^p \leq C \sum_{j=1}^n E \|V_j\|^p$$

for all independent \mathcal{X} -valued random elements V_1, \dots, V_n with mean 0.

We refer the reader to Pisier [10] and Woyczyński [16] for a detailed discussion of this notion.

Scalora [14] introduced the idea of the conditional expectation of a random element in a Banach space. For a random element V and sub σ -algebra \mathcal{G} of \mathcal{F} , the conditional expectation $E(V|\mathcal{G})$ is defined analogously to that in the random variable case and enjoys similar properties.

A real separable Banach space \mathcal{X} is said to be *martingale type* p ($1 \leq p \leq 2$) if there exists a finite positive constant C such that for all martingales $\{S_n; n \geq 1\}$ with values in \mathcal{X} ,

$$\sup_{n \geq 1} E \|S_n\|^p \leq C \sum_{n=1}^{\infty} E \|S_n - S_{n-1}\|^p.$$

It can be shown using classical methods from martingale theory that if \mathcal{X} is of martingale type p , then for all $1 \leq r < \infty$ there exists a finite constant C such that

$$E \sup_{n \geq 1} \|S_n\|^r \leq CE \left(\sum_{n=1}^{\infty} \|S_n - S_{n-1}\|^p \right)^{\frac{r}{p}}.$$

Clearly every real separable Banach space is of martingale type 1 and the real line (the same as any Hilbert space) is of martingale type 2. If a real separable Banach space of martingale type p for some $1 < p \leq 2$ then it is of martingale type r for all $r \in [1, p)$.

It follows from the Hoffmann-Jørgensen and Pisier [4] characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p , then it is of Rademacher type p . But the notion of martingale type p is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness. For proofs and more details, the reader may refer to Pisier [10, 11].

Definition. Let $0 < p \leq 2$. We say that a collection $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ of mn random elements taking values in a real separable Banach space \mathcal{X} satisfies the *maximal Marcinkiewicz-Zygmund inequality with exponent* p if

$$E \max_{\substack{1 \leq k \leq m \\ 1 \leq l \leq n}} \left\| \sum_{i=1}^k \sum_{j=1}^l V_{ij} \right\|^p \leq C \sum_{i=1}^m \sum_{j=1}^n E \|V_{ij}\|^p, \quad (2.1)$$

where the constant C is independent of m and n .

It is clear that for $0 < p \leq 1$, every collection random elements $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p . Now we provide an example of a collection of random elements $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ taking values in a real separable Banach space \mathcal{X} that satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p in the case of $1 < p \leq 2$.

Example. Suppose that $\{V_j; 1 \leq j \leq n\}$ is a collection of n independent mean 0 random elements taking values in a Rademacher type p ($1 < p \leq 2$) Banach space. Then there exists a constant C such that

$$E \left\| \sum_{j=1}^n V_j \right\|^p \leq C \sum_{j=1}^n E \|V_j\|^p.$$

This follows from the following observation. If we set $V_{1j} = V_j$ and $V_{ij} = 0$ for all $2 \leq i \leq m$, $1 \leq j \leq n$, then $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p by the definition of the Rademacher type p .

Random elements $\{V_{mn}; m \geq 1, n \geq 1\}$ are said to be *stochastically dominated* by a random element V if for some constant $C < \infty$

$$P\{\|V_{mn}\| > t\} \leq CP\{\|V\| > t\}, \quad t \geq 0, \quad m \geq 1, \quad n \geq 1.$$

Denote d_k be the number of divisors of k . The following lemma can be found in Gut and Spătaru [7] or Gut [8].

Lemma 1.

$$\sum_{k=1}^n \frac{d_k}{k^\gamma} = O(n^{1-\gamma} \log n) \quad (\gamma < 1), \tag{2.2}$$

$$\sum_{k=i+1}^{\infty} \frac{d_k}{k^\gamma} = O\left(\frac{\log i}{(i+1)^{\gamma-1}}\right) \quad (\gamma > 1). \tag{2.3}$$

3. MAIN RESULTS

With the preliminaries accounted for, the main results may now be established. In the following we let $\{V_{mn}; m \geq 1, n \geq 1\}$ be an array of random elements in a real separable Banach space \mathcal{X} .

Theorem 1. *Let $0 < p \leq 2$ and $\{V_{ij}; i \geq 1, j \geq 1\}$ be an array of random elements that satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p . If*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|V_{ij}\|^p}{(i^\alpha j^\beta)^p} < \infty$$

for some $\alpha > 0$ and $\beta > 0$, then

$$\frac{1}{m^\alpha n^\beta} \sum_{i=1}^m \sum_{j=1}^n V_{ij} \rightarrow 0 \quad \text{a.s. as } m \vee n \rightarrow \infty \tag{3.1}$$

and

$$\frac{1}{m^\alpha n^\beta} \sum_{i=1}^m \sum_{j=1}^n V_{ij} \rightarrow 0 \quad \text{in } L_p \quad \text{as } m \vee n \rightarrow \infty. \tag{3.2}$$

Proof. Set

$$S_{mn} = \sum_{i=1}^m \sum_{j=1}^n V_{ij},$$

and

$$T_{kl} = \max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \left\| \frac{S_{mn}}{m^\alpha n^\beta} - \frac{S_{2^k 2^l}}{2^{k\alpha} 2^{l\beta}} \right\|.$$

First, we prove (3.1). For arbitrary $\varepsilon > 0$, by the Markov and maximal Marcinkiewicz-Zygmund inequalities with exponent p we have

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P \left\{ \left\| \frac{S_{2^k 2^l}}{2^{k\alpha} 2^{l\beta}} \right\| > \varepsilon \right\} &\leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2^{k\alpha} 2^{l\beta})^{p\varepsilon^p}} E\|S_{2^k 2^l}\|^p \\ &\leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{C}{(2^{k\alpha} 2^{l\beta})^{p\varepsilon^p}} \sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E\|V_{ij}\|^p \leq \frac{C}{\varepsilon^p} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|V_{ij}\|^p}{(i^\alpha j^\beta)^p} < \infty. \end{aligned}$$

It follows by the Borel-Cantelli lemma that

$$\lim_{k \vee l \rightarrow \infty} \frac{S_{2^k 2^l}}{2^{k\alpha} 2^{l\beta}} = 0 \quad \text{a.s.} \tag{3.3}$$

Again, let $\varepsilon > 0$ be arbitrary, from the maximal Marcinkiewicz-Zygmund inequality with exponent p we obtain

$$\begin{aligned} P\{|T_{kl}| > \varepsilon\} &\leq P\left\{\frac{\|S_{2^k 2^l}\|}{2^{k\alpha} 2^{l\beta}} > \frac{\varepsilon}{2}\right\} + P\left\{\max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \frac{\|S_{mn}\|}{m^\alpha n^\beta} > \frac{\varepsilon}{2}\right\} \leq P\left\{\frac{\|S_{2^k 2^l}\|}{2^{k\alpha} 2^{l\beta}} > \frac{\varepsilon}{2}\right\} \\ &+ P\left\{\max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \frac{\|S_{mn}\|}{2^{k\alpha} 2^{l\beta}} > \frac{\varepsilon}{2}\right\} \leq P\left\{\|S_{2^k 2^l}\| > \frac{2^{k\alpha} 2^{l\beta} \varepsilon}{2}\right\} + P\left\{\max_{\substack{1 \leq m \leq 2^{k+1} \\ 1 \leq n \leq 2^{l+1}}} \|S_{mn}\| > \frac{2^{k\alpha} 2^{l\beta} \varepsilon}{2}\right\} \\ &\leq \frac{2^p}{(2^{k\alpha} 2^{l\beta})^p \varepsilon^p} E\|S_{2^k 2^l}\|^p + \frac{2^p}{(2^{k\alpha} 2^{l\beta})^p \varepsilon^p} E \max_{\substack{1 \leq m \leq 2^{k+1} \\ 1 \leq n \leq 2^{l+1}}} \|S_{mn}\|^p \leq \frac{2^p C}{(2^{k\alpha} 2^{l\beta})^p \varepsilon^p} \sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E\|V_{ij}\|^p \\ &+ \frac{2^p C}{(2^{k\alpha} 2^{l\beta})^p \varepsilon^p} \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} E\|V_{ij}\|^p \leq \frac{C}{\varepsilon^p} \sum_{i=1}^{2^k} \sum_{j=1}^{2^l} \frac{E\|V_{ij}\|^p}{(2^{k\alpha} 2^{l\beta})^p} + \frac{C}{\varepsilon^p} \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} \frac{E\|V_{ij}\|^p}{(2^{(k+1)\alpha} 2^{(l+1)\beta})^p} \\ &\leq \frac{C}{\varepsilon^p} \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} \frac{E\|V_{ij}\|^p}{(2^{(k+1)\alpha} 2^{(l+1)\beta})^p}. \end{aligned}$$

From this we obtain

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\{|T_{kl}| > \varepsilon\} \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{C}{\varepsilon^p} \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} \frac{E\|V_{ij}\|^p}{(2^{(k+1)\alpha} 2^{(l+1)\beta})^p} \leq \frac{C}{\varepsilon^p} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|V_{ij}\|^p}{(i^\alpha j^\beta)^p} < \infty.$$

Again by the Borel-Cantelli lemma, we have that

$$\lim_{k \vee l \rightarrow \infty} T_{kl} = 0 \quad \text{a.s.} \tag{3.4}$$

Note that for $2^k \leq m < 2^{k+1}$ and $2^l \leq n < 2^{l+1}$,

$$\left\| \frac{S_{mn}}{m^\alpha n^\beta} \right\| \leq \left\| \frac{S_{mn}}{m^\alpha n^\beta} - \frac{S_{2^k 2^l}}{2^{k\alpha} 2^{l\beta}} \right\| + \left\| \frac{S_{2^k 2^l}}{2^{k\alpha} 2^{l\beta}} \right\| \leq T_{kl} + \frac{\|S_{2^k 2^l}\|}{2^{k\alpha} 2^{l\beta}}, \tag{3.5}$$

and so the conclusion (3.1) follows from (3.3) and (3.4).

Next, we will prove (3.2). Since convergence of the following series

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2^{k\alpha} 2^{l\beta})^p} E\|S_{2^k 2^l}\|^p \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{C}{(2^{k\alpha} 2^{l\beta})^p} \sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E\|V_{ij}\|^p \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|V_{ij}\|^p}{(i^\alpha j^\beta)^p} < \infty,$$

we get

$$\lim_{k \vee l \rightarrow \infty} E\left\| \frac{S_{2^k 2^l}}{2^{k\alpha} 2^{l\beta}} \right\|^p = 0. \tag{3.6}$$

On the other hand, we have that

$$\begin{aligned} E|T_{kl}|^p &\leq CE \max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \left\| \frac{S_{mn}}{m^\alpha n^\beta} \right\|^p + CE \frac{\|S_{2^k 2^l}\|^p}{(2^{k\alpha} 2^{l\beta})^p} \leq \frac{C}{(2^{k\alpha} 2^{l\beta})^p} E \max_{\substack{1 \leq m \leq 2^{k+1} \\ 1 \leq n \leq 2^{l+1}}} \|S_{mn}\|^p \\ &+ \frac{C}{(2^{k\alpha} 2^{l\beta})^p} E\|S_{2^k 2^l}\|^p \leq \frac{C}{(2^{k\alpha} 2^{l\beta})^p} \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} E\|V_{ij}\|^p + \frac{C}{(2^{k\alpha} 2^{l\beta})^p} \sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E\|V_{ij}\|^p \end{aligned}$$

$$\leq C \sum_{i=1}^{2^k} \sum_{j=1}^{2^l} \frac{E\|V_{ij}\|^p}{(2^{k\alpha}2^{l\beta})^p} + C \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} \frac{E\|V_{ij}\|^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p} \leq C \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} \frac{E\|V_{ij}\|^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p}.$$

This implies

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} E|T_{kl}|^p \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} C \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} \frac{E\|V_{ij}\|^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p} \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|V_{ij}\|^p}{(i^\alpha j^\beta)^p} < \infty.$$

Hence we have

$$\lim_{k \vee l \rightarrow \infty} E|T_{kl}|^p = 0. \tag{3.7}$$

It follows from (3.5) that for $2^k \leq m < 2^{k+1}$, $2^l \leq n < 2^{l+1}$,

$$E \left\| \frac{S_{mn}}{m^\alpha n^\beta} \right\|^p \leq CE|T_{kl}|^p + CE \left\| \frac{S_{2^k 2^l}}{(2^{k\alpha}2^{l\beta})} \right\|^p.$$

Thus, the conclusion (3.2) follows from (3.6) and (3.7). □

In the next two theorems, we obtain the Marcinkiewicz-Zygmund type law of large numbers for double array of random elements.

Theorem 2. *Let $1 < r < p \leq 2$ and $\{V_{mn}; m \geq 1, n \geq 1\}$ be an array of random elements that satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p . Suppose that $\{V_{mn}; m \geq 1, n \geq 1\}$ is stochastically dominated by a random element V such that $E\|V\|^r \log^+ \|V\|^r < \infty$ if $1 \leq r < p$ then*

$$\frac{1}{(mn)^{\frac{1}{r}}} \sum_{i=1}^m \sum_{j=1}^n V_{ij} \rightarrow 0 \quad \text{a.s. and in } L_p \text{ as } m \vee n \rightarrow \infty. \tag{3.8}$$

Proof. Let F be the distribution of $\|V\|$. Set

$$V'_{ij} = V_{ij}I(\|V_{ij}\| \leq (ij)^{\frac{1}{r}}), V''_{ij} = V_{ij}I(\|V_{ij}\| > (ij)^{\frac{1}{r}}).$$

Applying the equation (2.3) of Lemma 2.1 with $\gamma = \frac{p}{r}$ we obtain $\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{p}{r}}} = O(\frac{\log i}{(i+1)^{\frac{p}{r}-1}})$. Then we have

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|V'_{ij}\|^p}{(ij)^{\frac{p}{r}}} &\leq C \sum_{k=1}^{\infty} \frac{d_k}{k^{\frac{p}{r}}} \int_0^{k^{\frac{1}{r}}} x^p dF(x) \leq C \sum_{i=0}^{\infty} \left(\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{p}{r}}} \right) \int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} x^p dF(x) \\ &\leq C \sum_{i=0}^{\infty} \frac{\log i}{(i+1)^{\frac{p}{r}-1}} \int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} x^p dF(x) \\ &\leq CE\|V\|^r \log^+ \|V\|^r < \infty. \end{aligned} \tag{3.9}$$

On the other hand, applying the equation (2.2) of Lemma 2.1 with $\gamma = \frac{1}{r}$ we obtain $\sum_{k=1}^n \frac{d_k}{k^{\frac{1}{r}}} = O(n^{1-\frac{1}{r}} \log n)$. Hence we have the following inequalities

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|(V''_{ij})\|}{(ij)^{\frac{1}{r}}} \leq C \sum_{k=1}^{\infty} \frac{d_k}{k^{\frac{1}{r}}} \int_{i^{\frac{1}{r}}}^{\infty} |x| dF(x) \leq C \sum_{k=1}^{\infty} \left(\sum_{i=1}^k \frac{d_i}{i^{\frac{1}{r}}} \right) \int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} |x| dF(x)$$

$$\leq C \sum_{i=1}^{\infty} i^{1-\frac{1}{r}} \log i \int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} |x| dF(x) \leq C \int_1^{\infty} |x|^r \log^+ |x|^r dF(x) \leq CE \|V\|^r \log^+ \|V\|^r < \infty.$$

This implies that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|(V''_{ij})\|^p}{(ij)^{\frac{2}{r}}} < \infty. \tag{3.10}$$

(3.9) and (3.10) yield

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|(V_{ij})\|^p}{(ij)^{\frac{2}{r}}} < \infty.$$

Applying Theorem 3.1 with $\alpha = \beta = \frac{1}{r}$ we obtain (3.8). □

Theorem 3. *Let $0 < r < 1$. If $\{V_{mn}; m \geq 1, n \geq 1\}$ is be stochastically dominated by a random element V such that $E\|V\|^r \log^+ \|V\|^r < \infty$, then*

$$\frac{1}{(mn)^{\frac{1}{r}}} \sum_{i=1}^m \sum_{j=1}^n V_{ij} \rightarrow 0 \text{ a.s. and in } L_1 \text{ as } m \vee n \rightarrow \infty. \tag{3.11}$$

Proof. By (2.3) of Lemma 2.1 with $\gamma = \frac{1}{r}$ we obtain $\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{1}{r}}} = O\left(\frac{\log i}{(i+1)^{\frac{1}{r}-1}}\right)$. Hence we can show that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|V_{ij}\|}{(ij)^{\frac{1}{r}}} \leq E\|V\|^r \log^+ \|V\|^r < \infty.$$

Applying Theorem 3.1 with $\alpha = \beta = \frac{1}{r}$ we obtain (3.11). □

In the next sections we present corollaries to the theorems.

4. MARTINGALE TYPE p BANACH SPACE CASE

The fact the a collection of a martingale difference random elements $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ taking values in martingale type p satisfy the maximal Marcinkiewicz-Zygmund inequality with exponent p , follows from the lemma below. We denote \mathcal{F}_{kl} the σ -field generated by the family of random elements $\{V_{ij}; i < k \text{ or } j < l\}$, $\mathcal{F}_{1,1} = \{\emptyset; \Omega\}$.

Lemma 2. *Let $1 \leq p \leq 2$ and let $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ be a collection of mn random elements in a real separable martingale type p Banach space with $E(V_{ij}|\mathcal{F}_{ij}) = 0$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Then $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p .*

Proof. The conclusion (2.1) is trivial in the case of $p = 1$. Hence we consider the case of $1 < p \leq 2$. Set $S_{kl} = \sum_{i=1}^k \sum_{j=1}^l V_{ij}$, $Y_l = \max_{1 \leq k \leq m} \|S_{kl}\|$. If σ_l is a σ -field generated by $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq l\}$ then for each $l(1 \leq l \leq n)$, $\sigma_l \subset \mathcal{F}_{i,l+1}$ for all $i \geq 1$. This implies that $E(V_{i,l+1}|\sigma_l) = E(E(V_{i,l+1}|\mathcal{F}_{i,l+1})|\sigma_l) = 0$.

Thus, we have

$$E(S_{k,l+1}|\sigma_l) = E(S_{kl}|\sigma_l) + \sum_{i=1}^k E(V_{i,l+1}|\sigma_l) = S_{kl},$$

that is, $\{S_{kl}, \sigma_l; 1 \leq l \leq n\}$ is a martingale. Hence, $\{\|S_{kl}\|, \sigma_l; 1 \leq l \leq n\}$ is a nonnegative submartingale for each $k = 1, 2, \dots, m$. By Lemma 2.2 of Thanh [13] it follows that $\{Y_l, \sigma_l; 1 \leq l \leq n\}$ is a nonnegative submartingale.

Applying Doob's inequality (see, e.g., Chow and Teicher [1, p. 255]), we obtain

$$E \left(\max_{\substack{1 \leq k \leq m \\ 1 \leq l \leq n}} \|S_{kl}\|^p \right) = E \left(\max_{1 \leq l \leq n} Y_l \right)^p \leq CEY_n^p. \tag{2.2}$$

On the other hand, since $E(V_{ij}|\mathcal{F}_{ij}) = 0$ we have that $\{S_{kn}, \mathcal{G}_k = \mathcal{F}_{k+1,1}; 1 \leq k \leq m\}$ is a martingale. Thus

$$EY_n^p = E \max_{1 \leq k \leq m} \|S_{kn}\|^p \leq C \sum_{k=1}^m E \left\| \sum_{j=1}^n V_{kj} \right\|^p. \tag{2.3}$$

Note again that for each $k(1 \leq k \leq m)$, $\{\sum_{j=1}^l V_{kj}, \mathcal{G}_{kl} = \mathcal{F}_{k,l+1}; 1 \leq l \leq n\}$ is a martingale. Hence,

$$E \left\| \sum_{j=1}^n V_{kj} \right\|^p \leq E \max_{1 \leq l \leq n} \left\| \sum_{j=1}^l V_{kj} \right\|^p \leq C \sum_{l=1}^n E \|V_{kl}\|^p. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) we obtain (2.1). □

The following theorem characterizes the martingale type p Banach spaces.

Theorem 4. *Let $\{V_{mn}; m \geq 1, n \geq 1\}$ be an array of random elements in a real separable Banach space \mathcal{X} . Then the following two statements are equivalent:*

(i) *The Banach \mathcal{X} is of martingale type p .*

(ii) *For every double arrays $\{V_{mn}; m \geq 1, n \geq 1\}$ of random elements in \mathcal{X} with $E(V_{mn}|\mathcal{F}_{mn}) = 0$ for all $m \geq 1, n \geq 1$, the condition*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E \|V_{mn}\|^p}{(m^\alpha n^\beta)^p} < \infty$$

for some $\alpha > 0$ and $\beta > 0$, implies that

$$\frac{1}{m^\alpha n^\beta} \sum_{i=1}^m \sum_{j=1}^n V_{ij} \rightarrow 0 \quad \text{a.s. as } m \vee n \rightarrow \infty.$$

Proof. First we prove $[(i) \Rightarrow (ii)]$.

The case of $1 < r < p \leq 2$ follows from Lemma 5.1 and Theorem 3.1. Hence we only prove for the case of $r = 1$. Applying (2.3) from Lemma 2.1 with $\gamma = p$, we obtain $\sum_{k=i+1}^{\infty} \frac{d_k}{k^p} = O\left(\frac{\log i}{(i+1)^{p-1}}\right)$. Hence we have

$$\begin{aligned} & \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E \|V'_{ij} - E(V'_{ij}|\mathcal{F}_{ij})\|^p}{(ij)^p} \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E \|V'_{ij}\|^p}{(ij)^p} \leq C \sum_{k=1}^{\infty} \frac{d_k}{k^p} \int_0^k x^p dF(x) \\ & \leq C \sum_{i=0}^{\infty} \left(\sum_{k=i+1}^{\infty} \frac{d_k}{k^p} \right) \int_i^{(i+1)} x^p dF(x) \leq C \sum_{i=0}^{\infty} \frac{\log i}{(i+1)^{p-1}} \int_i^{(i+1)} x^p dF(x) \leq CE \|V\| \log^+ \|V\| < \infty. \end{aligned}$$

By Theorem 3.1 with $\alpha = \beta = 1$

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (V'_{ij} - E(V'_{ij}|\mathcal{F}_{ij})) \rightarrow 0 \quad \text{a.s. as } m \vee n \rightarrow \infty.$$

Next, for arbitrary $\varepsilon > 0$,

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(\|V''_{ij}\| > \varepsilon) &\leq \sum_{k=1}^{\infty} d_k P(\|V\| > k) = \sum_{i=1}^{\infty} \left(\sum_{k=1}^i d_k \right) \int_i^{i+1} dF(x) \\ &\leq C \sum_{i=1}^{\infty} i \log i \int_i^{i+1} dF(x) \leq CE \|V\| \log^+ \|V\| < \infty. \end{aligned}$$

Hence by the Borel-Cantelli lemma

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n V''_{ij} \rightarrow 0 \quad \text{a.s.}$$

Finally, by the identity

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n V_{ij} = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (V'_{ij} - E(V'_{ij}|\mathcal{F}_{ij})) + \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n V''_{ij} + \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n E(V''_{ij}|\mathcal{F}_{ij}),$$

it is enough to show that

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (E(V''_{ij}|\mathcal{F}_{ij})) \rightarrow 0 \quad \text{a.s. as } m \vee n \rightarrow \infty. \tag{4.4}$$

The proof of (4.4) is the same as that of $\sum_{i=1}^m \sum_{j=1}^n [E(X'_{ij}|\mathcal{F}_{ij}) - E(X''_{ij}|\mathcal{F}_{ij})] \rightarrow 0$ a.s. as $m \vee n \rightarrow \infty$ of Hong and Volodin for $p = 1$ (pp. 1139–1140) with noting that

$$\|E(V''_{ij}|\mathcal{F}_{ij})\| \leq E(\|V''_{ij}\||\mathcal{F}_{ij}) \leq 2E(\|V\|I(\|V\| > ij)|\mathcal{F}_{ij}),$$

and we use $\|V''_{ij}\|$ and $\|V\|$ are instead of $|X''_{ij}|$ and $|X|$, respectively.

Now we prove $[(ii) \Rightarrow (i)]$. Assume that (ii) holds. Let $\{W_n, \mathcal{G}_n; n \geq 1\}$ be an arbitrary sequence of martingale difference in \mathcal{X} such that

$$\sum_{n=1}^{\infty} \frac{E\|W_n\|^p}{n^p} < \infty$$

For $n \geq 1$, set

$$V_{mn} = W_n \quad \text{if } m = 1 \quad \text{and} \quad V_{mn} = 0 \quad \text{if } m \geq 2.$$

Then $\{V_{mn}; m \geq 1, n \geq 1\}$ is a double array of random elements in \mathcal{X} satisfies $E(V_{mn}|\mathcal{F}_{mn}) = 0$ for all $m \geq 1, n \geq 1$, and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E\|V_{mn}\|^p}{(mn)^p} = \sum_{n=1}^{\infty} \frac{E\|W_n\|^p}{n^p} < \infty.$$

By (ii) ,

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n V_{ij} \rightarrow 0 \quad \text{a.s. as } m \vee n \rightarrow \infty.$$

Taking $m = 1$ and letting $n \rightarrow \infty$ we obtain

$$\frac{1}{n} \sum_{j=1}^n W_j \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.$$

Then by Theorem 2.2 of Hoffmann-Jørgensen and Pisier [4], \mathcal{X} is of martingale type p . □

The next corollary follows from Theorems 3.2 and 3.3.

Corollary 1. Let $1 \leq r < p \leq 2$ and \mathcal{X} be a martingale type p Banach space. Suppose that $\{V_{mn}; m \geq 1, n \geq 1\}$ is stochastically dominated by a random element V such that $E\|V\|^r \log^+ \|V\|^r < \infty$ if $1 \leq r < p$

$$\frac{1}{(mn)^{\frac{1}{r}}} \sum_{i=1}^m \sum_{j=1}^n (V_{ij} - E(V_{ij}|\mathcal{F}_{ij})) \rightarrow 0 \quad \text{a.s. and in } L_r \text{ as } m \vee n \rightarrow \infty.$$

5. RADEMACHER TYPE p BANACH SPACE CASE

The fact that a collection $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ of mn independent mean zero random elements spaces taking values in Rademacher type p satisfy the maximal Marcinkiewicz-Zygmund inequality with exponent p , follows from the following lemma. For the proof we refer to Lemma 2.3. of Thanh [13], while it is obvious.

Lemma 3. Let $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ be a collection of mn independent mean 0 random elements in a real separable Rademacher type p Banach spaces. Then $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p .

The following theorem can be proved in the same way as Theorem 4.2 and hence we omit its proof.

Theorem 5. Let $\{V_{mn}; m \geq 1, n \geq 1\}$ be an array of independent random elements in a real separable Banach space \mathcal{X} . Then the following two statements are equivalent:

- (i) The Banach \mathcal{X} is of Rademacher type p .
- (ii) For every double arrays $\{V_{mn}; m \leq 1, n \leq 1\}$ of independent mean 0 random elements in \mathcal{X} , the condition

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|V_{ij}\|^p}{(i^\alpha j^\beta)^p} < \infty$$

for some $\alpha > 0$ and $\beta > 0$ implies that

$$\frac{1}{m^\alpha n^\beta} \sum_{i=1}^m \sum_{j=1}^n V_{ij} \rightarrow 0 \quad \text{a.s. as } m \vee n \rightarrow \infty.$$

Note that the above result is more general than Theorem 3.1 (necessity part) of Rosalsky and Thanh [11].

An open problem. A relatively interesting case $r = 1$ is not considered in Theorem 3.2, while we propose that the result remains true in this case.

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