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On the almost sure growth rate of sums of lower negatively dependent nonnegative random variables

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Abstract

For a sequence of lower negatively dependent nonnegative random variables $\{X_n, n \geq 1\}$, conditions are provided under which $\lim_{n \rightarrow \infty} \sum_{j=1}^n X_j / b_n = \infty$ almost surely where $\{b_n, n \geq 1\}$ is a nondecreasing sequence of positive constants. The results are new even when they are specialized to the case of nonnegative independent and identically distributed summands and $b_n = n^r, n \geq 1$ where $r > 0$.

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1. Introduction

Throughout this paper, let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables defined on a probability space (Ω, \mathcal{F}, P) . Their partial sums will be denoted, as usual, by $S_n = \sum_{j=1}^n X_j, n \geq 1$. It is well known that if the random variables $\{X_n, n \geq 1\}$ are independent and identically

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distributed (i.i.d.) with $EX_1 = \infty$, then

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = \infty \quad \text{almost surely (a.s.).}$$

In this paper, we study the almost sure growth rate of S_n ; more specifically, we provide conditions under which

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = \infty \quad \text{a.s.,}$$

where $\{b_n, n \geq 1\}$ is a nondecreasing sequence of positive constants. In fact, we examine this problem in more generality than the case of i.i.d. summands. In the main results, Theorems 1 and 2, the summands $\{X_n, n \geq 1\}$ do not need to be independent or identically distributed but, nevertheless, they are new results in the i.i.d. case. In Theorems 1 and 2, it is assumed that $\{X_n, n \geq 1\}$ is a sequence of random variables which are *lower negatively dependent*. This is the assertion that for all $n \geq 1$ and all $x_1, \dots, x_n \in \mathbb{R}$,

$$P\{X_1 \leq x_1, \dots, X_n \leq x_n\} \leq \prod_{j=1}^n P\{X_j \leq x_j\}. \quad (1.1)$$

If for all $n > m \geq 1$ and all $x_1, x_2 \in \mathbb{R}$

$$P\{X_m \leq x_1, X_n \leq x_2\} \leq P\{X_m \leq x_1\} \cdot P\{X_n \leq x_2\},$$

then $\{X_n, n \geq 1\}$ is said to be a sequence of *pairwise lower negatively dependent* random variables. Of course, (1.1) is automatic if the $\{X_n, n \geq 1\}$ are independent. A sequence of lower negatively dependent random variables $\{Y_n, n \geq 1\}$ (not necessarily nonnegative) obeys the *strong law of large numbers* (SLLN) $\sum_{j=1}^n Y_j / b_n \rightarrow 0$ a.s. under suitable conditions; see for example Matuła (1992), Kim and Baek (1999) (wherein $b_n = n$, $n \geq 1$), Amini and Bozorgnia (2000) (wherein $b_n = n^r$, $n \geq 1$ with $r > \frac{1}{2}$), Kim and Kim (2001), and Taylor et al. (2002). In Kim and Kim (2001) the norming sequence $\{b_n, n \geq 1\}$ is very rapidly growing in that it satisfies the condition $\sum_{j=n}^{\infty} b_j^{-2} = \mathcal{O}(b_n^{-2})$ and for this reason their assertion that their Theorem 1 extends Theorem 6 of Adler et al. (1992) (from the independent case to the pairwise lower negatively dependent case) is incorrect.

2. Mainstream

The key lemma for proving Theorem 1 will now be established.

Lemma 1. *Let $\{S_n, n \geq 1\}$ be a nondecreasing sequence of nonnegative random variables, let $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive constants, and let $\{a_n, n \geq 1\}$ be a sequence of positive constants. Suppose that there exists a strictly increasing sequence of positive integers $\{m(k), k \geq 1\}$ and a constant $M < \infty$ such that*

$$b_{m(k+1)} \leq M b_{m(k)}, \quad k \geq 1 \quad (2.1)$$

and for some $\varepsilon > 0$

$$\min_{m(k)+1 \leq n \leq m(k+1)} a_n \geq \frac{\varepsilon}{m(k+1) - m(k)}, \quad k \geq 1. \tag{2.2}$$

If for some $0 < c_0 \leq \infty$ and all $0 < c < c_0$

$$\sum_{n=1}^{\infty} a_n P\{S_n \leq cb_n\} < \infty, \tag{2.3}$$

then

$$\liminf_{n \rightarrow \infty} \frac{S_n}{b_n} \geq \frac{c_0}{M^2} \quad \text{a.s.} \tag{2.4}$$

Proof. For arbitrary $0 < c < c_0$,

$$\begin{aligned} &\infty > \sum_{k=1}^{\infty} \sum_{n=m(k)+1}^{m(k+1)} a_n P\{S_n \leq cb_n\} \quad (\text{by (2.3)}) \\ &\geq \sum_{k=1}^{\infty} \frac{\varepsilon}{m(k+1) - m(k)} P\{S_{m(k+1)} \leq cb_{m(k)}\} (m(k+1) - m(k)) \\ &\quad (\text{by (2.2), } S_n \uparrow, \text{ and } b_n \uparrow) \\ &\geq \varepsilon \sum_{k=1}^{\infty} P\left\{S_{m(k+1)} \leq \frac{c}{M} b_{m(k+1)}\right\} \quad (\text{by (2.1)}) \end{aligned}$$

and hence

$$\sum_{k=1}^{\infty} P\{S_{m(k)} \leq cb_{m(k)}\} < \infty$$

for all $0 < c < \frac{c_0}{M}$. Then by the Borel–Cantelli lemma

$$P\{S_{m(k)} \leq cb_{m(k)} \text{ i.o. } (k)\} = 0$$

for all $0 < c < \frac{c_0}{M}$. Hence

$$\liminf_{k \rightarrow \infty} \frac{S_{m(k)}}{b_{m(k)}} \geq c \quad \text{a.s.}$$

for all $0 < c < \frac{c_0}{M}$. Letting $c \uparrow \frac{c_0}{M}$ yields

$$\liminf_{k \rightarrow \infty} \frac{S_{m(k)}}{b_{m(k)}} \geq \frac{c_0}{M} \quad \text{a.s.}$$

Then for $n \geq m(1) + 1$, writing $m(k) + 1 \leq n \leq m(k + 1)$ where $k = k(n) \geq 1$ and recalling that $0 \leq S_n \uparrow, 0 < b_n \uparrow$, and (2.1), we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{S_n}{b_n} &\geq \liminf_{n \rightarrow \infty} \frac{S_{m(k(n))}}{b_{m(k(n)+1)}} \\ &\geq \liminf_{n \rightarrow \infty} \frac{S_{m(k(n))}}{M b_{m(k(n))}} \\ &\geq \liminf_{k \rightarrow \infty} \frac{S_{m(k)}}{M b_{m(k)}} \\ &\geq \frac{c_0}{M^2} \quad \text{a.s.} \quad \square \end{aligned}$$

Corollary 1. *Let $\{S_n, n \geq 1\}$ be a nondecreasing sequence of nonnegative random variables and let $r > 0$. If for some $0 < c_0 \leq \infty$ and all $0 < c < c_0$*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leq cn^r\} < \infty,$$

then

$$\liminf_{n \rightarrow \infty} \frac{S_n}{n^r} \geq c_0 \quad \text{a.s.}$$

Proof. Let $\delta > 0$ and $\theta > 1$ be arbitrary and let $m(k) = [\theta^k], k \geq 1$. Let $b_n = n^r, n \geq 1$. Set $M = (1 + \delta)\theta^r$. Then $b_{m(k+1)} \leq M b_{m(k)}$ for all large k . Set $a_n = 1/n, n \geq 1$. Now for some $\varepsilon > 0$ and large k

$$\min_{m(k)+1 \leq n \leq m(k+1)} a_n \geq \frac{\varepsilon}{m(k+1) - m(k)}.$$

It is clear that the sequences $\{b_n, n \geq 1\}, \{m(k), k \geq 1\}$, and $\{a_n, n \geq 1\}$ can be redefined for small values of n and k so that (2.1), (2.2), and (2.3) (for all $0 < c < c_0$) hold with M and ε as above. Hence by Lemma 1

$$\liminf_{n \rightarrow \infty} \frac{S_n}{n^r} \geq \frac{c_0}{(1 + \delta)^2 \theta^{2r}} \quad \text{a.s.}$$

Letting $\delta \downarrow 0$ and $\theta \downarrow 1$ yields the conclusion (2.4). \square

The next corollary of Lemma 1 was originally due to Gut et al. (1997) when $S_n = \sum_{j=1}^n X_j, n \geq 1$ where $\{X_n, n \geq 1\}$ is a sequence of i.i.d. nonnegative random variables and $b_n = n^r, n \geq 1$ with $r \geq 1$ but the proof of it provided by them is incorrect. However, a valid proof of their result can be given using the method of proof of another result by Gut et al. (1997). It should be noted that the faster $b_n \uparrow \infty$ in Corollary 2, the stronger is the assumption (2.6) but so is the conclusion (2.7).

Corollary 2. *Let $\{S_n, n \geq 1\}$ be a nondecreasing sequence of nonnegative random variables and let $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive constants such that*

$$b_{2n} = \mathcal{O}(b_n). \tag{2.5}$$

If

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leq cb_n\} < \infty \tag{2.6}$$

for all $0 < c < \infty$, then

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = \infty \quad a.s. \tag{2.7}$$

Proof. Let $m(k) = 2^k, k \geq 1, \varepsilon = \frac{1}{2}$, and $a_n = 1/n, n \geq 1$. It follows from (2.5) that (2.1) holds for some constant $M < \infty$. Moreover, (2.2) is immediate. Taking $c_0 = \infty$, the conclusion (2.7) follows directly from Lemma 1. \square

Theorem 1 may now be presented. Its proof was inspired by that of a classical result of **Derman and Robbins (1955)** showing that for i.i.d. summands $\{Y_n, n \geq 1\}$ with $EY_1^+ = EY_1^- = \infty$ that $\lim_{n \rightarrow \infty} \sum_{j=1}^n Y_j/n = \infty$ a.s. can prevail.

Theorem 1. Let $S_n = \sum_{j=1}^n X_j, n \geq 1$ where $\{X_n, n \geq 1\}$ is a sequence of lower negatively dependent nonnegative random variables. Let $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive constants and let $\{a_n, n \geq 1\}$ be a sequence of positive constants. Suppose that there exists a strictly increasing sequence of positive integers $\{m(k), k \geq 1\}$ such that $b_{m(k+1)} = \mathcal{O}(b_{m(k)})$ and (2.2) holds for some $\varepsilon > 0$. Furthermore, suppose that for all $0 < c < \infty$

$$\sum_{n=1}^{\infty} a_n \prod_{j=1}^n P\{X_j \leq cb_n\} < \infty. \tag{2.8}$$

Then

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = \infty \quad a.s. \tag{2.9}$$

Proof. Since $b_{m(k+1)} = \mathcal{O}(b_{m(k)})$, there exists a constant $M < \infty$ such that (2.1) holds. Let $0 < c < \infty$ be arbitrary and set $c_0 = \infty$. Then for $n \geq 1$,

$$\begin{aligned} P\{S_n \leq cb_n\} &\leq P\{X_1 \leq cb_n, \dots, X_n \leq cb_n\} \text{ (since the } X_j \text{ are nonnegative)} \\ &\leq \prod_{j=1}^n P\{X_j \leq cb_n\} \text{ (by (1.1))} \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} a_n P\{S_n \leq cb_n\} \leq \sum_{n=1}^{\infty} a_n \prod_{j=1}^n P\{X_j \leq cb_n\} < \infty \text{ (by (2.8)).}$$

The conclusion (2.9) follows immediately from Lemma 1 noting that $0 \leq S_n \uparrow$. \square

The next theorem is in effect a special case of Theorem 1. Theorem 2 can of course also be proved in a similar manner to that of Theorem 1 by using Corollary 2 instead of Lemma 1.

Theorem 2. Let $S_n = \sum_{j=1}^n X_j, n \geq 1$ where $\{X_n, n \geq 1\}$ is a sequence of lower negatively dependent nonnegative random variables and suppose that the nonnegative function $h(x)$ on $[0, \infty)$ is such that

$$P\{X_n > x\} \geq h(x) \quad \text{for all } n \geq 1 \text{ and } x \geq 0. \tag{2.10}$$

Let $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive constants satisfying (2.5). Suppose that for all $0 < c < \infty$

$$\sum_{n=1}^{\infty} \frac{(1 - h(cb_n))^n}{n} < \infty. \tag{2.11}$$

Then

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = \infty \quad \text{a.s.} \tag{2.12}$$

Proof. Let $m(k) = 2^k, k \geq 1, \varepsilon = \frac{1}{2}$, and $a_n = 1/n, n \geq 1$. Then $b_{m(k+1)} = \mathcal{O}(b_{m(k)})$ and (2.2) holds as was noted in the proof of Corollary 2. Note that for all $0 < c < \infty$

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \prod_{j=1}^n P\{X_j \leq cb_n\} &= \sum_{n=1}^{\infty} \frac{1}{n} \prod_{j=1}^n (1 - P\{X_j > cb_n\}) \\ &\leq \sum_{n=1}^{\infty} \frac{(1 - h(cb_n))^n}{n} \quad (\text{by (2.10)}) \\ &< \infty \quad (\text{by (2.11)}). \end{aligned}$$

The conclusion (2.12) follows directly from Theorem 1. \square

We now obtain the following corollary of Theorem 2. It should be noted that (2.13) and (2.14) ensure that $EX_n^{1/r} = \infty, n \geq 1$. Moreover, the larger r is taken in Corollary 3, the more stringent are the hypotheses but the conclusion (2.15) is also stronger.

Corollary 3. Let $S_n = \sum_{j=1}^n X_j, n \geq 1$ where $\{X_n, n \geq 1\}$ is a sequence of lower negatively dependent nonnegative random variables. Let $r > 0$ and suppose there exists a nonnegative function $g(x)$ on $[0, \infty)$ such that

$$x^{1/r} P\{X_n > x\} \geq g(x) \quad \text{for all } n \geq 1 \text{ and } x \geq 0 \tag{2.13}$$

and

$$\log \log x = o(g(x)) \quad \text{as } x \rightarrow \infty. \tag{2.14}$$

Then

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^r} = \infty \quad \text{a.s.} \tag{2.15}$$

Proof. We will apply Theorem 2 with the function $h(x)$ defined by $h(0) = 0, h(x) = g(x)/x^{1/r}, x > 0$ and with $b_n = n^r, n \geq 1$. By (2.14) we can write $g(x) = G(x) \log \log x$ where $G(x) \rightarrow \infty$ as $x \rightarrow \infty$. For $0 < c < \infty$ and all large n

$$\begin{aligned} \frac{(1 - h(cn^r))^n}{n} &\leq \frac{\left(\exp\left\{-\frac{g(cn^r)}{(cn^r)^{1/r}}\right\}\right)^n}{n} \quad (\text{by the elementary inequality } 1 - x \leq e^{-x}) \\ &= \frac{1}{n} \exp\left\{-\frac{g(cn^r)}{c^{1/r}}\right\} \\ &= \frac{1}{n} \exp\left\{-\frac{G(cn^r) \log \log(cn^r)}{c^{1/r}}\right\} \\ &\leq \frac{1}{n} \exp\{-2 \log \log n\} \\ &= \frac{1}{n(\log n)^2} \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} \frac{(1 - h(cn^r))^n}{n} < \infty.$$

The conclusion (2.15) follows immediately from Theorem 2. \square

The fourth corollary is in effect a special case of Corollary 3.

Corollary 4. Let $S_n = \sum_{j=1}^n X_j, n \geq 1$ where $\{X_n, n \geq 1\}$ is a sequence of identically distributed lower negatively dependent (a fortiori, i.i.d.) nonnegative random variables. Let $r > 0$ and suppose that

$$\frac{x^{1/r} P\{X_1 > x\}}{\log \log x} \rightarrow \infty \quad \text{as } x \rightarrow \infty. \tag{2.16}$$

Then

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^r} = \infty \quad \text{a.s.} \tag{2.17}$$

Proof. Let $g(x) = x^{1/r} P\{X_1 > x\}, x \geq 0$. Then (2.13) holds since the $\{X_n, n \geq 1\}$ are identically distributed, and (2.14) holds by (2.16). The conclusion (2.17) then follows immediately from Corollary 3. \square

The following example demonstrates that Corollary 4 is sharp and hence so are Corollary 3 and Theorems 1 and 2.

Example 1. Let $r \geq 1$ and let $\{X_n, n \geq 1\}$ be a sequence of identically distributed lower negatively dependent random variables where

$$P\{X_1 > x\} = \frac{e^{e/r} (\log \log x)^2}{x^{1/r}}, \quad x \geq e^e.$$

Then (2.16) holds whence (2.17) follows from Corollary 4. We will now show that for arbitrary $\varepsilon > 0$, the SLLN

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^{r+\varepsilon}} = 0 \quad \text{a.s.} \tag{2.18}$$

holds. Note that

$$\begin{aligned} EX_1^{1/(r+\varepsilon)} &= \text{Const.} + \int_{e^e}^{\infty} e^{e/r} x^{\frac{1}{r+\varepsilon}-1} \frac{(\log \log x)^2}{x^{1/r}} dx \\ &= \text{Const.} + \int_{e^e}^{\infty} e^{e/r} \frac{(\log \log x)^2}{x^{1+\frac{1}{r}-\frac{1}{r+\varepsilon}}} dx \\ &< \infty. \end{aligned} \tag{2.19}$$

It is well known (see, e.g., Sawyer, 1966; Chatterji, 1969/1970; or Martikainen and Petrov, 1980) that the famous Marcinkiewicz–Zygmund SLLN holds irrespective of the joint distributions of the identically distributed summands $\{Y_n, n \geq 1\}$ when $E|Y_1|^p < \infty$, where $0 < p < 1$. Hence (2.18) follows from (2.19) and the Marcinkiewicz–Zygmund SLLN.

Remark 1. Suppose that $\{X_n, n \geq 1\}$ is a sequence of i.i.d. nonnegative random variables with

$$P\{X_1 > x\} \sim \frac{c \log \log x}{x^{1/r}} v(x) \quad \text{as } x \rightarrow \infty, \tag{2.20}$$

where c and r are positive constants and $v(x)$ is a positive function. If $v(x) \rightarrow \infty$ as $x \rightarrow \infty$, then $S_n/n^r \rightarrow \infty$ a.s. by Corollary 4. But if $v(x) \rightarrow 0$ as $x \rightarrow \infty$, then $\liminf_{n \rightarrow \infty} S_n/n^r < \infty$ a.s. can prevail. This follows by taking $v(x) = (\log \log x)^{-1/r}$, $x \geq e^e$ where $r > 1$ and applying Theorem 5 of Erickson (1976) which establishes that $\liminf_{n \rightarrow \infty} S_n/n^r = b$ a.s. for some constant $0 < b < \infty$.

Remark 2. Suppose that $\{X_n, n \geq 1\}$ is a sequence of i.i.d. nonnegative random variables satisfying (2.20) with $v(x) \equiv 1$. If $0 < r \leq 1$, then $\lim_{n \rightarrow \infty} S_n/n^r = \infty$ a.s. by the Kolmogorov SLLN. An interesting question which we are unable to resolve is whether or not $\lim_{n \rightarrow \infty} S_n/n^r = \infty$ a.s. when $r > 1$. However, it follows from the ensuing theorem that $\liminf_{n \rightarrow \infty} S_n/n^r \geq c^r$ a.s. It is interesting to notice that the lower bound c^r depends on the distribution of X_1 . In addition, since $\sum_{n=1}^{\infty} P\{X_n > Mn^r\} = \infty$ for all $0 < M < \infty$, it follows from the Borel–Cantelli lemma that

$$\limsup_{n \rightarrow \infty} \frac{S_n}{n^r} \geq \limsup_{n \rightarrow \infty} \frac{X_n}{n^r} = \infty \quad \text{a.s.}$$

Theorem 3. Let $S_n = \sum_{j=1}^n X_j$, $n \geq 1$ where $\{X_n, n \geq 1\}$ is a sequence of identically distributed lower negatively dependent (a fortiori, i.i.d.) nonnegative random variables. Suppose that

$$P\{X_1 > x\} \geq \frac{C(x) \log \log x}{x^{1/r}}, \quad x > e, \tag{2.21}$$

where $\lim_{x \rightarrow \infty} C(x) = c \in (0, \infty)$ and $r > 0$. Then

$$\liminf_{n \rightarrow \infty} \frac{S_n}{n^r} \geq c^r \quad a.s. \tag{2.22}$$

Proof. Let $c_0 = c^r$. Now for all $0 < \gamma < c_0$,

$$\lim_{n \rightarrow \infty} \frac{C(\gamma n^r)}{\gamma^{1/r}} = \frac{c}{\gamma^{1/r}} > 1$$

and so $C(\gamma n^r)/\gamma^{1/r} \geq 1 + \delta$ for some $\delta > 0$ and large n . Thus for all $0 < \gamma < c_0$ and all large n

$$\begin{aligned} P\{S_n \leq \gamma n^r\} &\leq P\{X_1 \leq \gamma n^r, \dots, X_n \leq \gamma n^r\} \\ &\quad (\text{since the } X_j \text{ are nonnegative}) \\ &\leq \left(1 - \frac{C(\gamma n^r) \log \log(\gamma n^r)}{\gamma^{1/r} n}\right)^n \quad (\text{by (1.1) and (2.21)}) \\ &\leq \exp\left\{-\frac{C(\gamma n^r)}{\gamma^{1/r}} \log \log(\gamma n^r)\right\} \\ &\quad \left(\text{since } \left(1 - \frac{x}{n}\right)^n \leq e^{-x} \text{ for } 0 < x < n\right) \\ &= (\log(\gamma n^r))^{-C(\gamma n^r)/\gamma^{1/r}} \\ &\leq (\log(\gamma n^r))^{-1-\delta} \leq \frac{2}{r^{1+\delta}(\log n)^{1+\delta}}. \end{aligned}$$

Consequently,

$$\sum_{n=1}^{\infty} a_n P\{S_n \leq \gamma n^r\} < \infty.$$

The conclusion (2.22) follows immediately from Corollary 1. \square

Remark 3. Corollary 4 may be contrasted with the following result of Rosalsky (1993) concerning the growth rate of sums of identically distributed strictly positive random variables irrespective of their joint distributions.

Theorem 4 (Rosalsky, 1993). Let $S_n = \sum_{j=1}^n X_j, n \geq 1$ where $\{X_n, n \geq 1\}$ is a sequence of identically distributed random variables with $X_1 > 0$ a.s. and let $\{b_n, n \geq 1\}$ be a sequence of positive constants with $\liminf_{n \rightarrow \infty} (b_n/n) = 0$. Then

$$\limsup_{n \rightarrow \infty} \frac{S_n}{b_n} = \infty \quad a.s.$$

irrespective of the joint distributions of the $\{X_n, n \geq 1\}$.

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