

On Complete Convergence for Arrays of Random Elements

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Abstract: A complete convergence theorem for arrays of rowwise independent random variables was obtained by Kruglov, Volodin, and Hu (*Statistics and Probability Letters* 2006, 76:1631–1640). In this article, we extend the result to a Banach space without any additional conditions. No assumptions are made concerning the geometry of the underlying Banach space.

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1. INTRODUCTION

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [1] as follows. A sequence

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$\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty \quad \text{for all } \epsilon > 0.$$

In view of the Borel–Cantelli lemma, this implies that $U_n \rightarrow \theta$ almost surely. The converse is true if $\{U_n, n \geq 1\}$ are independent random variables. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdős [2] proved the converse.

The result of Hsu–Robbins–Erdős has been generalized and extended in several directions. Some of these generalizations are in a Banach space setting. A sequence of Banach space valued random elements is said to converge completely to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

Rowwise independence means that the random elements within each row are independent but that no independence is assumed between rows.

Recently, Kruglov et al. [3] proved the following complete convergence theorem for arrays of rowwise independent random variables $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$, where $\{k_n, n \geq 1\}$ is a sequence of positive integers.

Theorem 1. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random variables and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that the following conditions hold:*

- (i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \epsilon) < \infty$ for all $\epsilon > 0$.
- (ii) there exist $J > 0, \delta > 0$, and $p \geq 1$ such that

$$\sum_{n=1}^{\infty} a_n \left(E \left| \sum_{i=1}^{k_n} (X_{ni} I(|X_{ni}| \leq \delta) - EX_{ni} I(|X_{ni}| \leq \delta)) \right|^p \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P \left(\max_{1 \leq m \leq k_n} \left| \sum_{i=1}^m (X_{ni} - EX_{ni} I(|X_{ni}| \leq \delta)) \right| > \epsilon \right) < \infty \quad \text{for all } \epsilon > 0.$$

In Theorem 1, J is not necessary an integer. In this article, we extend Theorem 1 to a separable Banach space without any additional conditions. No assumptions are made concerning the geometry of the underlying Banach space.

2. MAIN RESULTS

To prove our main results, we will need the following lemma. It was proved in Hu et al. [4] as a version of the famous Hoffmann–Jørgensen inequality for independent, but not necessarily symmetric, random variables.

Lemma 1. *If X_1, \dots, X_n are independent random variables, then for every integer $j \geq 1$ and $t > 0$*

$$P(|S_n| > 6^j t) \leq C_j P\left(\max_{1 \leq i \leq n} |X_i| > \frac{t}{4^{j-1}}\right) + D_j \max_{1 \leq i \leq n} \left[P\left(|S_i| > \frac{t}{4^j}\right) \right]^{2^j},$$

where C_j and D_j are positive constants depending only on j , and $S_i = \sum_{l=1}^i X_l$ for $1 \leq i \leq n$.

Note that Lemma 1 is still valid for independent, but not necessarily symmetric, random elements.

Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent, but not necessarily identically distributed, random elements taking values in a separable Banach space B . In general the case $k_n = \infty$ is not being precluded. In this case we are assuming that the series of random elements in Theorems 2(ii) and 3(ii) converges almost surely.

Theorem 2. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that the following conditions hold:*

- (i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty$ for all $\epsilon > 0$.
- (ii) there exist $J > 0$ and $\delta > 0$ such that

$$\sum_{n=1}^{\infty} a_n \left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta) - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\|^J \right) < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P\left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^m (X_{ni} - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\| > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$ be given. Without loss of generality, we may assume $0 < \epsilon \leq \delta$. We proceed with two cases.

Case 1: $J \geq 1$.

Take a positive integer j such that $2^j \geq J$. Define $X'_{ni} = X_{ni} I(\|X_{ni}\| \leq \epsilon/(4^{j-1} \cdot 16 \cdot 6^j))$ and $X''_{ni} = X_{ni} I(\epsilon/(4^{j-1} \cdot 16 \cdot 6^j) < \|X_{ni}\| \leq \delta)$ for $1 \leq$

$i \leq k_n$ and $n \geq 1$. The set of all natural numbers is partitioned into two subsets

$$A' = \left\{ n : \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon / (4^{j-1} \cdot 16 \cdot 6^j)) \leq 1 \right\}$$

and

$$A'' = \left\{ n : \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon / (4^{j-1} \cdot 16 \cdot 6^j)) > 1 \right\}.$$

Applying (i) we obtain

$$\sum_{n \in A''} a_n \leq \sum_{n \in A''} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon / (4^{j-1} \cdot 16 \cdot 6^j)) < \infty.$$

By the inequality of Etemadi [5], we have that

$$\begin{aligned} &P\left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^m (X_{ni} - EX_{ni}I(\|X_{ni}\| \leq \delta)) \right\| > \epsilon\right) \\ &\leq 4 \max_{1 \leq m \leq k_n} P\left(\left\| \sum_{i=1}^m (X_{ni} - EX_{ni}I(\|X_{ni}\| \leq \delta)) \right\| > \frac{\epsilon}{4}\right). \end{aligned}$$

Hence it is enough to show that

$$\sum_{n \in A'} a_n \max_{1 \leq m \leq k_n} P\left(\left\| \sum_{i=1}^m (X_{ni} - EX_{ni}I(\|X_{ni}\| \leq \delta)) \right\| > \frac{\epsilon}{4}\right) < \infty.$$

Observe that

$$\begin{aligned} &\sum_{n \in A'} a_n \max_{1 \leq m \leq k_n} P\left(\left\| \sum_{i=1}^m (X_{ni} - EX_{ni}I(\|X_{ni}\| \leq \delta)) \right\| > \frac{\epsilon}{4}\right) \\ &\leq \sum_{n \in A'} a_n \max_{1 \leq m \leq k_n} P\left(\left\| \sum_{i=1}^m (X'_{ni} - EX'_{ni}) \right\| > \frac{\epsilon}{8}\right) \\ &\quad + \sum_{n \in A'} a_n \max_{1 \leq m \leq k_n} P\left(\left\| \sum_{i=1}^m (X''_{ni} - EX''_{ni}) \right\| > \frac{\epsilon}{8}\right) \\ &\quad + \sum_{n \in A'} a_n \max_{1 \leq m \leq k_n} \sum_{i=1}^m P(\|X_{ni}\| > \delta) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For I_1 , we note that

$$\max_{1 \leq i \leq m} \|X'_{ni} - EX'_{ni}\| \leq \epsilon / (4^{j-1} \cdot 8 \cdot 6^j).$$

By Lemma 1, we obtain

$$\begin{aligned}
 & \max_{1 \leq m \leq k_n} P\left(\left\|\sum_{i=1}^m (X'_{ni} - EX'_{ni})\right\| > \frac{\epsilon}{8}\right) \\
 & \leq C_j \max_{1 \leq m \leq k_n} P\left(\max_{1 \leq i \leq m} \|X'_{ni} - EX'_{ni}\| > \frac{\epsilon}{4^{j-1} \cdot 8 \cdot 6^j}\right) \\
 & \quad + D_j \max_{1 \leq m \leq k_n} \max_{1 \leq i \leq m} \left[P\left(\left\|\sum_{l=1}^i (X'_{nl} - EX'_{nl})\right\| > \frac{\epsilon}{4^j \cdot 8 \cdot 6^j}\right) \right]^{2j} \\
 & \leq D_j \max_{1 \leq i \leq k_n} \left[P\left(\left\|\sum_{l=1}^i (X'_{nl} - EX'_{nl})\right\| > \frac{\epsilon}{4^j \cdot 8 \cdot 6^j}\right) \right]^J \\
 & \leq D_j \left(\frac{4^j \cdot 8 \cdot 6^j}{\epsilon}\right)^J \max_{1 \leq i \leq k_n} \left(E \left\|\sum_{l=1}^i (X'_{nl} - EX'_{nl})\right\|\right)^J \\
 & = D_j \left(\frac{4^j \cdot 8 \cdot 6^j}{\epsilon}\right)^J \left(E \left\|\sum_{i=1}^{k_n} (X'_{ni} - EX'_{ni})\right\|\right)^J.
 \end{aligned}$$

The last equality follows by Lemma 3 in Etemadi [6]. By the triangle inequality, we have

$$\begin{aligned}
 & E \left\|\sum_{i=1}^{k_n} (X'_{ni} - EX'_{ni})\right\| \\
 & = E \left\|\sum_{i=1}^{k_n} (X_{ni}I(\|X_{ni}\| \leq \delta) - EX_{ni}I(\|X_{ni}\| \leq \delta)) - \sum_{i=1}^{k_n} (X''_{ni} - EX''_{ni})\right\| \\
 & \leq E \left\|\sum_{i=1}^{k_n} (X_{ni}I(\|X_{ni}\| \leq \delta) - EX_{ni}I(\|X_{ni}\| \leq \delta))\right\| + E \left\|\sum_{i=1}^{k_n} (X''_{ni} - EX''_{ni})\right\| \\
 & \leq E \left\|\sum_{i=1}^{k_n} (X_{ni}I(\|X_{ni}\| \leq \delta) - EX_{ni}I(\|X_{ni}\| \leq \delta))\right\| + 2 \sum_{i=1}^{k_n} E \|X''_{ni}\| \\
 & \leq E \left\|\sum_{i=1}^{k_n} (X_{ni}I(\|X_{ni}\| \leq \delta) - EX_{ni}I(\|X_{ni}\| \leq \delta))\right\| \\
 & \quad + 2\delta \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon/(4^{j-1} \cdot 16 \cdot 6^j)).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 I_1 \leq D_j \left(\frac{4^j \cdot 8 \cdot 6^j}{\epsilon}\right)^J \sum_{n \in A'} a_n \left(E \left\|\sum_{i=1}^{k_n} (X_{ni}I(\|X_{ni}\| \leq \delta) - EX_{ni}I(\|X_{ni}\| \leq \delta))\right\| \right. \\
 \left. + 2\delta \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon/(4^{j-1} \cdot 16 \cdot 6^j))\right)^J
 \end{aligned}$$

$$\begin{aligned}
 &\leq D_j \left(\frac{4^j \cdot 8 \cdot 6^j}{\epsilon} \right)^J 2^{J-1} \\
 &\quad \times \sum_{n \in A'} a_n \left[\left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta)) - EX_{ni} I(\|X_{ni}\| \leq \delta) \right\| \right)^J \right. \\
 &\quad \quad \left. + \left(2\delta \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon/(4^{j-1} \cdot 16 \cdot 6^j)) \right)^J \right] \\
 &\leq D_j \left(\frac{4^j \cdot 8 \cdot 6^j}{\epsilon} \right)^J 2^{J-1} \\
 &\quad \times \sum_{n \in A'} a_n \left[\left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta)) - EX_{ni} I(\|X_{ni}\| \leq \delta) \right\| \right)^J \right. \\
 &\quad \quad \left. + (2\delta)^J \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon/(4^{j-1} \cdot 16 \cdot 6^j)) \right],
 \end{aligned}$$

since $n \in A'$ and $J \geq 1$. Thus $I_1 < \infty$ by (i) and (ii).

For I_2 , we have by Markov's inequality that

$$\begin{aligned}
 I_2 &\leq \frac{8}{\epsilon} \sum_{n \in A'} a_n \max_{1 \leq m \leq k_n} E \left\| \sum_{i=1}^m (X''_{ni} - EX''_{ni}) \right\| \\
 &\leq \frac{16}{\epsilon} \sum_{n \in A'} a_n \sum_{i=1}^{k_n} E \|X''_{ni}\| \\
 &\leq \frac{16\delta}{\epsilon} \sum_{n \in A'} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon/(4^{j-1} \cdot 16 \cdot 6^j)).
 \end{aligned}$$

Thus $I_2 < \infty$ by (i). Obviously $I_3 < \infty$ by (i).

Case 2: $0 < J < 1$.

The set of all natural numbers is partitioned into two subsets

$$B' = \left\{ n : E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta)) - EX_{ni} I(\|X_{ni}\| \leq \delta) \right\| \leq 1 \right\}$$

and

$$B'' = \left\{ n : E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta)) - EX_{ni} I(\|X_{ni}\| \leq \delta) \right\| > 1 \right\}.$$

Applying (ii) we obtain

$$\sum_{n \in B''} a_n \leq \sum_{n \in B''} a_n \left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta)) - EX_{ni} I(\|X_{ni}\| \leq \delta) \right\| \right)^J < \infty.$$

Hence it is enough to show that

$$\sum_{n \in B'} a_n P \left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^m (X_{ni} - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\| > \epsilon \right) < \infty.$$

For $1 \leq i \leq k_n$ and $n \geq 1$, define

$$Y_{ni} = \begin{cases} X_{ni} & \text{if } n \in B', \\ 0 & \text{if } n \in B''. \end{cases}$$

Then $\{Y_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of rowwise independent random elements. Further, we have by (i) and (ii) that

$$\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|Y_{ni}\| > \epsilon) < \infty \quad \text{for all } \epsilon > 0$$

and

$$\sum_{n=1}^{\infty} a_n E \left\| \sum_{i=1}^{k_n} (Y_{ni} I(\|Y_{ni}\| \leq \delta) - EY_{ni} I(\|Y_{ni}\| \leq \delta)) \right\| < \infty.$$

Applying Case 1 (with $J = 1$) to the array $\{Y_{ni}, 1 \leq i \leq k_n, n \geq 1\}$, we obtain

$$\sum_{n=1}^{\infty} a_n P \left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^m (Y_{ni} - EY_{ni} I(\|Y_{ni}\| \leq \delta)) \right\| > \epsilon \right) < \infty \quad \text{for all } \epsilon > 0.$$

But

$$\begin{aligned} & \sum_{n=1}^{\infty} a_n P \left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^m (Y_{ni} - EY_{ni} I(\|Y_{ni}\| \leq \delta)) \right\| > \epsilon \right) \\ &= \sum_{n \in B'} a_n P \left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^m (X_{ni} - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\| > \epsilon \right). \end{aligned}$$

Hence

$$\sum_{n \in B'} a_n P \left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^m (X_{ni} - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\| > \epsilon \right) < \infty. \quad \square$$

By using Theorem 2, we can obtain the following theorem which extends Theorem 1 to a Banach space without any additional conditions.

Theorem 3. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements and $\{a_n, n \geq 1\}$ a sequence of positive*

constants. Suppose that the following conditions hold:

- (i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty$ for all $\epsilon > 0$.
 (ii) there exist $J > 0$, $\delta > 0$, and $p \geq 1$ such that

$$\sum_{n=1}^{\infty} a_n \left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta) - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\|^p \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P \left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^m (X_{ni} - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\| > \epsilon \right) < \infty \text{ for all } \epsilon > 0.$$

Proof. By (ii), we have that

$$\begin{aligned} & \sum_{n=1}^{\infty} a_n \left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta) - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\|^p \right)^{pJ} \\ & \leq \sum_{n=1}^{\infty} a_n \left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta) - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\|^p \right)^J < \infty. \end{aligned}$$

The result follows from Theorem 2 with J replaced by pJ . \square

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