

MARCINKIEWICZ-TYPE LAW OF LARGE NUMBERS FOR DOUBLE ARRAYS

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ABSTRACT. Chatterji strengthened version of a theorem for martingales which is a generalization of a theorem of Marcinkiewicz proving that if X_n is a sequence of independent, identically distributed random variables with $E|X_n|^p < \infty$, $0 < p < 2$ and $EX_1 = 0$ if $1 \leq p < 2$ then $n^{-1/p} \sum_{i=1}^n X_i \rightarrow 0$ a.s. and in L^p . In this paper, we prove a version of law of large numbers for double arrays. If $\{X_{ij}\}$ is a double sequence of random variables with $E|X_{11}|^p \log^+ |X_{11}|^p < \infty$, $0 < p < 2$, then $\lim_{m \vee n \rightarrow \infty} \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - a_{ij})}{(mn)^{\frac{1}{p}}} = 0$ a.s. and in L^p , where $a_{ij} = 0$ if $0 < p < 1$, and $a_{ij} = E[X_{ij} | \mathcal{F}_{ij}]$ if $1 \leq p \leq 2$, which is a generalization of Etemadi's Marcinkiewicz-type SLLN for double arrays. This also generalize earlier results of Smythe, and Gut for double arrays of i.i.d. r.v.'s.

Let N denote the set of positive integer. And we note \prec the lexicographic order on $N \times N$, i.e., $(i, j) \prec (k, l)$ if and only if either $i < k$ or $i = k$ and $j < l$. And let \mathcal{F}_{ij} be the σ -field generated by the family of random variables $\{X_{kl} | (k, l) \prec (i, j)\}$.

To prove the main theorem, we need the following lemma.

LEMMA. Let $\{X_{ij}\}$ be a double sequence of random variables with $E[X_{ij} | \mathcal{F}_{ij}] = 0$. Given any $\epsilon > 0$, we have

$$(1) \quad P \left\{ \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left| \sum_{k=1}^i \sum_{l=1}^j X_{kl} \right| > \epsilon \right\} \leq \frac{4}{\epsilon^2} \sum_{i=1}^m \sum_{j=1}^n EX_{ij}^2.$$

Received March 13, 1999.

1991 Mathematics Subject Classification: 60G50, 60F15.

Key words and phrases: strong law of large numbers, double arrays, L^p convergence.

Proof. Let $S_{mn} = \sum_{i=1}^m \sum_{j=1}^n X_{ij}$. And let $Y_j = \max\{|S_{ij}| : i = 1, \dots, m\}$, for each $j = 1, \dots, n$. If σ_l is a σ -field generated by $\{X_{ij} | 1 \leq i \leq m, 1 \leq j \leq l\}$,

$$\begin{aligned} E[S_{kl} | \sigma_{l-1}] &= E[S_{k,l-1} + X_{1l} + \dots + X_{kl} | \sigma_{l-1}] \\ &= E[S_{k,l-1} | \sigma_{l-1}] + E[X_{1l} | \sigma_{l-1}] + \dots + E[X_{kl} | \sigma_{l-1}] \\ &= S_{k(l-1)}. \end{aligned}$$

Since $E[X_{il} | \sigma_{l-1}] = E[E[X_{il} | \mathcal{F}_{ml}] | \sigma_{l-1}] = 0$ by hypothesis, which implies that $\{S_{kl}, \sigma_l\}_{l=1}^n$ is a martingale. Hence $\{|S_{kl}|, \sigma_l\}_{l=1}^n$ is a submartingale for each $k = 1, \dots, m$, which follows that $\{Y_l^2, \sigma_l\}$ is a nonnegative submartingale. Applying Theorem 3.3.4 and Corollary 3.3.2 [8], we obtain the inequalities

$$\begin{aligned} P \left\{ \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} |S_{ij}| > \epsilon \right\} &= P \left\{ \max_{1 \leq j \leq n} Y_j^2 > \epsilon^2 \right\} \\ &\leq \frac{1}{\epsilon} EY_n^2 \\ &\leq \frac{4}{\epsilon} \sum_{i=1}^m \sum_{j=1}^n EX_{ij}^2 \end{aligned}$$

which gives us the desired results. \square

THEOREM 1. Let $\{X_{ij}\}$ be a double sequence of random variables such that either $E|X|^p \log^+ |X|^p < \infty$, $0 < p < 2$, $p \neq 1$ and $P\{|X_{ij}| \geq x\} \leq P\{|X| \geq x\}$, $0 \leq x < \infty$ or $E|X| \log^+ |X| < \infty$ and $P(|X_{ij}| \geq x | \mathcal{F}_{ij}) \leq P(|X| \geq x | \mathcal{F}_{ij})$ a.s. Then

$$(2) \quad \lim_{m \vee n \rightarrow \infty} \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - a_{ij})}{mn^{\frac{1}{p}}} = 0 \quad \text{a.s.}$$

where $a_{ij} = 0$ if $0 < p < 1$, and $a_{ij} = E[X_{ij} | \mathcal{F}_{ij}]$ if $1 \leq p \leq 2$.

Proof. Let \mathcal{F} be the distribution function of X and let $X'_{ij} = X_{ij} I\{|X_{ij}| \leq (ij)^{\frac{1}{p}}\}$ with I the indicator function, $X''_{ij} = X_{ij} - X'_{ij}$, and let d_k be the number of divisors of k . Denote by $S_{mn} = \sum_{i=1}^m \sum_{j=1}^n X_{ij}$, $S'_{mn} = \sum_{i=1}^m \sum_{j=1}^n X'_{ij}$. By using the fact that $\sum_{k=1}^n d_k = O(n \log n)$, we obtain

the inequalities

$$\begin{aligned}
 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\{X_{ij} \neq X'_{ij}\} &\leq \sum_{k=1}^{\infty} d_k P\{|X| > k^{\frac{1}{p}}\} \\
 (3) \qquad \qquad \qquad &= \sum_{i=1}^{\infty} \left(\sum_{k=1}^i d_k \right) \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} dF(x) \\
 &\leq c \sum_{i=1}^{\infty} i \log i \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} dF(x) \\
 &\leq cE|X|^p \log^+ |X|^p < \infty,
 \end{aligned}$$

where c is an unimportant positive constant which is allowed to change. Hence by the Borel-Cantelli lemma

$$(4) \qquad \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - X'_{ij})}{(mn)^{\frac{1}{p}}} \Rightarrow 0 \quad \text{a.s.}$$

Since $I_d(\cdot) - E[\cdot|\mathcal{F}_{ij}]$ is a contraction on L^2 , we obtain

$$\begin{aligned}
 &\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P \left\{ \left| \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} (X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}])}{(2^k 2^l)^{\frac{1}{p}}} \right| > \epsilon \right\} \\
 &\leq c \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{E(\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} (X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}]))^2}{(2^k 2^l)^{\frac{2}{p}}} \\
 &= c \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E(X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}])^2}{(2^k 2^l)^{\frac{2}{p}}} \\
 &\leq c \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E(X'_{ij})^2}{(2^k 2^l)^{\frac{2}{p}}} \\
 &\leq c \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E(X'_{ij})^2}{(ij)^{\frac{2}{p}}}.
 \end{aligned}$$

If we use the fact that $\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{2}{p}}} = O\left(\frac{\log i}{(i+1)^{\frac{2}{p}-1}}\right)$,

$$\begin{aligned}
 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E(X'_{ij})^2}{(ij)^{\frac{2}{p}}} &\leq c \sum_{k=1}^{\infty} \frac{d_k}{k^{\frac{2}{p}}} \int_0^{k^{\frac{1}{p}}} x^2 dF(x) \\
 (5) \qquad \qquad \qquad &\leq c \sum_{i=0}^{\infty} \left(\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{2}{p}}} \right) \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} x^2 df(x) \\
 &\leq c \sum_{i=0}^{\infty} \frac{\log i}{(i+1)^{\frac{2}{p}-1}} \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} x^2 dF(x) \\
 &\leq cE|X|^p \log^+ |X|^p < \infty,
 \end{aligned}$$

which follows easily by summation by part. It follows that

$$(6) \qquad \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} (X'_{ij} - E[X'_{ij} | \mathcal{F}_{ij}])}{(2^k 2^l)^{\frac{1}{p}}} \rightarrow 0 \quad \text{a.s.}$$

And

$$\begin{aligned}
 T_{kl} &= \max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \left\| \frac{S_{2^k 2^l}^*}{(2^k 2^l)^{\frac{1}{p}}} - \frac{S_{mn}^*}{(mn)^{\frac{1}{p}}} \right\| \\
 &\leq \frac{|S_{2^k 2^l}^*|}{(2^k 2^l)^{\frac{1}{p}}} + \max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \frac{|S_{mn}^*|}{(mn)^{\frac{1}{p}}},
 \end{aligned}$$

where $S_{mn}^* = \sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - E[X'_{ij} | \mathcal{F}_{ij}])$. From Lemma we obtain, for any $\epsilon > 0$,

$$\begin{aligned}
 P\{|T_{kl}| > \epsilon\} &\leq P\left\{ \frac{|S_{2^k 2^l}^*|}{(2^k 2^l)^{\frac{1}{p}}} > \frac{\epsilon}{2} \right\} + P\left\{ \max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \frac{|S_{mn}^*|}{(mn)^{\frac{1}{p}}} > \frac{\epsilon}{2} \right\} \\
 (7) \qquad \qquad &\leq P\left\{ \frac{|S_{2^k 2^l}^*|}{(2^k 2^l)^{\frac{1}{p}}} > \frac{\epsilon}{2} \right\} + \frac{16}{\epsilon^2} \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} \frac{E|X'_{ij}|^2}{(2^k 2^l)^{\frac{2}{p}}}.
 \end{aligned}$$

Since $\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P\{|T_{kl}| > \epsilon\} < \infty$ from the inequalities in (5), by Borel-Cantelli lemma, we obtain

$$(8) \qquad \qquad \qquad T_{kl} \rightarrow 0 \quad \text{a.s.}$$

Hence (6) and (8) give us

$$(9) \quad \frac{\sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - E[X'_{ij} | \mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} \rightarrow 0 \quad \text{a.s.}$$

Combining (4) and (9), we get

$$(10) \quad \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - E[X'_{ij} | \mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} \rightarrow 0 \quad \text{a.s.}$$

Suppose that $1 < p < 2$. Since

$$\begin{aligned} \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - E[X_{ij} | \mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} &= \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - E[X'_{ij} | \mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} \\ &\quad - \frac{\sum_{i=1}^m \sum_{j=1}^n E[X''_{ij} | \mathcal{F}_{ij}]}{(mn)^{\frac{1}{p}}}, \end{aligned}$$

it remains to prove that the second term of right-hand side converges to 0. By Markov's inequality

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P \left\{ \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E[|X''_{ij}| | \mathcal{F}_{ij}]}{(2^k 2^l)^{\frac{1}{p}}} > \epsilon \right\} \leq \frac{1}{\epsilon} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E|X''_{ij}|}{(2^k 2^l)^{\frac{1}{p}}}.$$

If we use the fact that $\sum_{k=1}^n \frac{d_k}{k^{\frac{1}{p}}} = O(n^{1-\frac{1}{p}} \log n)$, we obtain the following inequalities

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E|X''_{ij}|}{(2^k 2^l)^{\frac{1}{p}}} &\leq c \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E|X'_{ij}|}{(ij)^{\frac{1}{p}}} \\ &\leq c \sum_{k=1}^{\infty} \frac{d_k}{k^{\frac{1}{p}}} \int_{k^{\frac{1}{p}}}^{\infty} |x| dF(x) \\ &= c \sum_{k=1}^{\infty} \left(\sum_{k=1}^i \frac{d_k}{k^{\frac{1}{p}}} \right) \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} |x| dF(x) \\ &\leq c \sum_{i=1}^{\infty} i^{1-\frac{1}{p}} \log i \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} |x| dF(x) \\ &\leq c \int_1^{\infty} |x|^p \log^+ |x|^p dF(x) \\ &\leq c E|X_{11}|^p \log^+ |X_{11}|^p < \infty. \end{aligned}$$

It follows that

$$(11) \quad \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E[|X''_{ij}| | \mathcal{F}_{ij}]}{(2^k 2^l)^{\frac{1}{p}}} \rightarrow 0 \quad \text{a.s.}$$

But

$$\begin{aligned} T'_{kl} &= \max_{\substack{2^k \leq m \leq 2^{k+1} \\ 2^l \leq n \leq 2^{l+1}}} \left| \frac{\sum_{i=1}^m \sum_{j=1}^n E[|X''_{ij}| | \mathcal{F}_{ij}]}{(mn)^{\frac{1}{p}}} - \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E[|X''_{ij}| | \mathcal{F}_{ij}]}{(2^k 2^l)^{\frac{1}{p}}} \right| \\ (12) \quad &\leq \frac{2^{\frac{2}{p}}}{(2^{k+1} 2^{l+1})^{\frac{1}{p}}} \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} E[|X''_{ij}| | \mathcal{F}_{ij}]. \end{aligned}$$

Since $\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} E|T'_{kl}| < \infty$, T'_{kl} converges to 0 which implies by (11)

$$\frac{\sum_{i=1}^m \sum_{j=1}^n E[|X''_{ij}| | \mathcal{F}_{ij}]}{(mn)^{\frac{1}{p}}} \rightarrow 0 \quad \text{a.s.}$$

Therefore the equation (2) holds for $1 < p < 2$. For $0 < p < 1$, by using the fact that $\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{1}{p}}} = O\left(\frac{\log i}{(i+1)^{\frac{1}{p-1}}}\right)$, we can show that

$$\frac{\sum_{i=1}^m \sum_{j=1}^n E[|X'_{ij}| | \mathcal{F}_{ij}]}{(mn)^{\frac{1}{p}}} \rightarrow 0 \quad \text{a.s.}$$

by the argument similar to the case $1 < p < 2$. Hence by (10)

$$\frac{\sum_{i=1}^m \sum_{j=1}^n X_{ij}}{(mn)^{\frac{1}{p}}} \rightarrow 0 \quad \text{a.s.}$$

Now for $p = 1$, we note that

$$E[X''_{ij} | \mathcal{F}_{ij}] \leq E[XI\{|X| > ij\} | \mathcal{F}_{ij}].$$

In the identity

$$\begin{aligned} & (mn)^{-1} \sum_{i=1}^m \sum_{j=1}^n (X_{ij} - E[X_{ij} | \mathcal{F}_{ij}]) \\ &= (mn)^{-1} \sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - E[X'_{ij} | \mathcal{F}_{ij}]) \\ & \quad + (mn)^{-1} \sum_{i=1}^m \sum_{j=1}^n [X''_{ij} + (E[X'_{ij} | \mathcal{F}_{ij}] - E[X_{ij} | \mathcal{F}_{ij}])], \end{aligned}$$

the first term on the right converges a.s. to 0 by (9). Since $(mn)^{-1} \sum_{i=1}^m \sum_{j=1}^n X''_{ij} \rightarrow 0$ a.s by (4), it will be enough to show that $(mn)^{-1} \sum_{i=1}^m \sum_{j=1}^n (E[X'_{ij} | \mathcal{F}_{ij}] - E[X_{ij} | \mathcal{F}_{ij}]) \rightarrow 0$ a.s. We note that by the stronger hypothesis for $p = 1$

$$\begin{aligned} |E[X'_{ij} | \mathcal{F}_{ij}] - E[X_{ij} | \mathcal{F}_{ij}]| &\leq E[|X''_{ij}| | \mathcal{F}_{ij}] \\ &\leq 2E[|X|I\{|X| > ij\} | \mathcal{F}_{ij}]. \end{aligned}$$

And using the fact that $|X|I\{|X| > n + 1\} < |X|I\{|X| > n\}$, we see that both $E[|X|I\{|X| > n\} | \mathcal{F}_{1n}]$, $n = 1, 2, \dots$ and $E[|X|I\{|X| > n\} | \mathcal{F}_{in}]$, $n = 1, 2, \dots$ are positive super-martingale. Since every positive super-martingale converges a.s., $\lim_{k \rightarrow \infty} E[|X|I\{|X| > k\} | \mathcal{F}_{1k}] = Y$ exist a.s. But $\int Y \leq \lim_{k \rightarrow \infty} E\{|X|I\{|X| > k\}\} = 0$ so that Y being non-negative must be zero a.s. and similarly $\lim_{k \rightarrow \infty} E[|X|I\{|X| > k\} | \mathcal{F}_{k1}] = 0$ a.s.

Now, noting that

$$\begin{aligned} E[|X|I\{|X| > ij\}|\mathcal{F}_{ik}] &\geq E[|X|I\{|X| > i(j+k)\}|\mathcal{F}_{i(j+k)}], \\ k = 0, 1, 2, \dots, \\ E[|X|I\{|X| > ij\}|\mathcal{F}_{ik}] &\geq E[|X|I\{|X| > (i+k)j\}|\mathcal{F}_{(i+k)j}], \\ k = 0, 1, 2, \dots, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{i=1}^m \sum_{j=1}^n E[|X''_{ij}|\mathcal{F}_{ij}] \\ &\leq \min \left\{ m \sum_{j=1}^n E[|X|I\{|X| > j\}|\mathcal{F}_{1j}], n \sum_{i=1}^m E[|X|I\{|X| > i\}|\mathcal{F}_{i1}] \right\}. \end{aligned}$$

If $m \vee n \rightarrow \infty$, then either $m \rightarrow \infty$ or $n \rightarrow \infty$. Without loss of generality we assume $n \rightarrow \infty$. Then

$$(mn)^{-1} \sum_{i=1}^m \sum_{j=1}^n E[|X''_{ij}|\mathcal{F}_{ij}] \leq \frac{1}{n} \sum_{j=1}^n E[|X|I\{|X| > j\}|\mathcal{F}_{1j}] \rightarrow 0 \quad \text{a.s.}$$

The theorem is thus completely proved. \square

THEOREM 2. *Under the same conditions of Theorem 1, we also have L^p -convergence.*

Proof. We first consider the case $1 < p < 2$. In the identity

$$\begin{aligned} &\frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - E[X_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} \\ &= \frac{\sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} + \frac{\sum_{i=1}^m \sum_{j=1}^n (X''_{ij} - E[X''_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}}, \end{aligned}$$

the first term on the right hand side converges in L^p . For

$$\begin{aligned} &\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{E\left(\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} (X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}])\right)^2}{(2^k 2^l)^{\frac{2}{p}}} \\ &\leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E(X'_{ij})^2}{(2^k 2^l)^{\frac{2}{p}}} < \infty \quad \text{by (5),} \end{aligned}$$

which implies $\frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} (X'_{ij} - E[X'_{ij} | \mathcal{F}_{ij}])}{(2^k 2^l)} \rightarrow 0$ in L^2 and

$$\begin{aligned} T'_{kl} &= \max_{\substack{2^k \leq m \leq 2^{k+1} \\ 2^l \leq n \leq 2^{l+1}}} \left| \frac{\sum_{i=1}^m \sum_{j=1}^n E(X'_{ij})^2}{(mn)^{\frac{2}{p}}} - \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E(X'_{ij})^2}{(2^k 2^l)^{\frac{2}{p}}} \right| \\ &\leq 4^{\frac{2}{p}} \frac{\sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} E(X'_{ij})^2}{(2^{k+1} 2^{l+1})^{\frac{2}{p}}} \rightarrow 0 \end{aligned}$$

by (5) again, which implies the first term converges to 0 in L^2 and hence in L^p . For the second term,

$$\begin{aligned} \frac{\sum_{i=1}^m \sum_{j=1}^n E|X''_{ij} - E[X''_{ij} | \mathcal{F}_{ij}]|^p}{mn} &\leq \frac{\alpha}{mn} \sum_{i=1}^m \sum_{j=1}^n E|X''_{ij}|^p \\ &\leq \frac{\alpha}{mn} m \sum_{j=1}^n E|X''_{1j}|^p \\ &= \frac{\alpha}{n} \sum_{j=1}^n E|X''_{1j}|^p \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where the first inequality comes from Essen-Von Bahr inequality. Consider now the case $0 < p < 1$ and note that

$$\frac{\sum_{i=1}^m \sum_{j=1}^n E|X_{ij}|}{(mn)^{\frac{1}{p}}} \leq \frac{\sum_{i=1}^m \sum_{j=1}^n E|X'_{ij}|}{(mn)^{\frac{1}{p}}} + \frac{\sum_{i=1}^m \sum_{j=1}^n E|X''_{ij}|}{(mn)^{\frac{2}{p}}}.$$

We already show that in the processes of the proof of Theorem 1 that the two terms on the right hand side converges 0.

The proof for $p = 1$ is as before except for one small detail. In the identity

$$\begin{aligned} &\frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - E[X_{ij} | \mathcal{F}_{ij}])}{(mn)} \\ &= \frac{\sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - E[X'_{ij} | \mathcal{F}_{ij}])}{mn} + \frac{\sum_{i=1}^m \sum_{j=1}^n (X''_{ij} - E[X''_{ij} | \mathcal{F}_{ij}])}{(mn)} \end{aligned}$$

the first term on the right converges in L^2 to 0 as before and the second converges to 0 in L^1 . Since

$$\begin{aligned} E \left| \frac{\sum_{i=1}^m \sum_{j=1}^n (X''_{ij} - E[X''_{ij} | \mathcal{F}_{ij}])}{mn} \right| &\leq \frac{\sum_{i=1}^m \sum_{j=1}^n E|X''_{ij} - E[X''_{ij} | \mathcal{F}_{ij}]|}{(mn)} \\ &\leq \frac{2 \sum_{i=1}^m \sum_{j=1}^n E|X''_{ij}|}{(mn)} \\ &\leq \frac{2m \sum_{j=1}^n E|X''_{1j}|}{(mn)} \\ &= \frac{2}{n} \sum_{j=1}^n E|X''_{1j}|, \end{aligned}$$

and the last term goes to 0 as $n \rightarrow \infty$, by the fact that $E|X_{1j}| \rightarrow 0$ as $j \rightarrow \infty$. The theorem is thus completed. \square

REMARK 1. The generalization to r dimensional array of random variables can be obtained under the condition

$$E|X|^p (\log^+ |X|^p)^{(r-1)} < \infty, \quad \text{for } 0 < p < 2.$$

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