Russian Mathematics (Iz.VUZ)
Vol.36, №4, pp.3-12, 1992

Izvestiya VUZ. Matematika
UDC 519.216

ON B_φ-CONVEX SPACES

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§ 1. Introduction

In the present article we generalize the notion of B-convex space and establish strengthened laws of large numbers for weighed sums of random elements with respect to such a generalization. The generalization's essence is concerned with the replacement of power function in the convexity definition by an arbitrary function φ which satisfies certain conditions on growth. Introduced B_φ-convex spaces are characterized by the laws of large numbers for weighted sums, while the known before convex spaces were characterized by the Marcinkiewicz laws of large numbers. We also establish conditions, under which the generalized space is of a stable type p.

Let us introduce notation. Henceforward $(E,\|.\|)$ means a real separable normed space, $B(E)=\{x\in E:\|x\|\le 1\}$ is the unit ball in E, $(X_k)_1^\infty$ is a sequence of independent symmetric random elements with their values in E, $S_n=\sum_{k=1}^{n}X_k$ and we denote by C certain constant quantity which can assume different values even within the same formula.

A sequence of random elements (X_k) is said to be *stochastically dominated* by a positive random variable ξ (notation is $(X_k)<\xi$) if there exists a $C>0$ such that $\sup_{k\ge 1}P(\|X_k\|>t)\le CP(\xi>t)$ for all $t>0$.

We recall the notions of Rademacher's and stable types of Banach spaces. We shall say that ε is a *Bernoulli random variable* if $P(\varepsilon=1)=P(\varepsilon=-1)=1/2$. Let (ε_k) be a sequence of independent Bernoulli random variables and (γ_k^p), $1\le p\le 2$, be a sequence of independent standard p-stable random variables with the characteristic function $\exp(-|t|^p)$.

If for any sequence $(x_k)\subset E$ the convergence of the series $\sum_{k=1}^{\infty}\|x_k\|^p$ implies the almost sure (a.s.) convergence of the series $\sum_{k=1}^{\infty}\gamma_k^p x_k$, then the Banach space E is said to possess a *stable type* p, $1\le p\le 2$. If for any sequence $(x_k)\subset E$ the convergence of the series $\sum_{k=1}^{\infty}\|x_k\|^p$ implies the a.s. convergence of the series $\sum_{k=1}^{\infty}\varepsilon_k x_k$, then the Banach space E is of *Rademacher type* p, $1\le p\le 2$. One can find additional information on the type p in [1] (ch.2, §2), [2] and [3] (ch. 3-4).

A few words about the history of the problem. A.Beck in [4] had introduced the notion of a B-convex Banach space and proved the strong law of large numbers, which characterizes these spaces. In [2], B.Maurey and G.Pisier studied the connection between the notion of B-convex space and this of the space of stable type 1. A.Shangua (see [5] and [6]) had introduced the notion of B_p-convex space, which is a generalization of the notion of B-convex space. The same author had given its characterization in terms of the Marcinkiewicz strong law of large numbers. M.Marcus and W.A.Woyczynski in [7] obtained as well a characterization of spaces of stable type p (which is similar to the Shangua's theorem, but the B_p-convexity had not been introduced) via the Marcinkiewicz strong law of large numbers. Let us dwell on this question in greater detail.

Assume that $1\leq p<2$. A Banach space E is said to be B_p*-convex* if there exist $\varepsilon>0$ and integer $k\geq 2$ both such that for all $x_1,\dots,x_k\in\mathbb{B}(E)$ it is possible to choose the signs "$\pm$" so that the inequality $\|\pm x_1\pm\dots\pm x_k\|\leq(1-\varepsilon)k^{1/p}$ will be valid.

THEOREM (M.Marcus, W.A.Woyczynski and A.Shangua). *Let* $1\leq p<2$. *The following statements are equivalent:*

(1) *E is of stable type p;*

(2) *for any independent centered random elements* (X_k) *such that* $(X_k)\prec\xi$ *and* $\mathbb{E}|\xi|^p<\infty$, *the sequence* $S_n/n^{1/p}\to 0$ *a.s.;*

(3) *E is a B_p-convex space.*

If we replace the condition of stochastic domination of (X_k) in the statement (2) of Marcus-Woyczynski-Shangua's Theorem by the condition of a common distribution of (X_k), then we obtain the characterization of spaces of Rademacher type p (see [8] and [9]).

Finally, let us note that T.Mikosch and R.Norvaisa ([10], Theorem 3.2) had considered the problem of characterization of spaces of type p via the convergence of weighted sums. These authors investigated the law of large numbers in the case of multidimensional index of summing. So, we have to reformulate slightly their result.

Let (w_n,a_n) be a sequences of real numbers such that $\lim\limits_{n\to\infty}a_n=\infty$ and $\lim\limits_{n\to\infty}w_n/a_n=0$. We introduce the class

$$F_p=\left\{(w_n,a_n):\sum_{n=k}^{\infty}n^{-p-1}w_n/a_n=O(k^{-p}w_k/a_k)\text{ with }k\to\infty\right\}.$$

We consider weighted sums of the form $Z_n=\frac{1}{a_n}\sum\limits_{k=1}^{n}w_kX_k$, where (X_k) are independent symmetric random elements.

THEOREM (T.Mikosch and R.Norvaisa). *Let* $1\leq p<2$ *and* $(w_n,a_n)\in F_q$ *for all* $q>p$. *The following statements are equivalent:*

(1) *E is of stable type p;*

(2) *for all independent symmetric random elements* (X_k) *such that* $(X_k)\prec\xi$ *and* $\mathbb{E}\xi^p<\infty$, *the sequence* $Z_n\to 0$ *a.s. with* $n\to\infty$;

(3) *for all independent symmetric random elements* (X_k) *such that* $(X_k)\prec\xi$ *and* $\mathbb{E}\xi^p<\infty$, *the sequence* $S_n/n^{1/p}\to 0$ *a.s.*

Let us dwell, in particular, on the implication (2)⇒(3). This is precisely the implication which, being absolutely evident, allows to conclude that (3)⇒(1) with the help of, e.g., (3)⇒(1) in Marcus-Woyczynski-Shangua Theorem. The implication (2)⇒(3) shows that in the item (2) the validity of law of large numbers is meant to be satisfied "for all at once" weights $(w_n,a_n)\in F_q$. The problem which we study is somewhat different: what one can say on a presence of the stable type, when the law of large numbers holds true at least for one weight? We shall give, by these means, an answer to our question posed at the end of [11].

§ 2. Definition of B_φ-convex spaces and their simplest properties

Let E be a normed space and $\varphi=(\varphi(k))_1^\infty$ be an increasing sequence of positive numbers.

Definition. Let ε be positive 0 and $k\geq 2$ be an integer. The space E is said to be (φ,k,ε)*-convex* if for all $x_i\in\mathbb{B}(E)$, $1\leq i\leq k$, one can choose the signs "$\pm$" so that the inequalities $\|\pm x_1\pm\dots\pm x_k\|\leq(1-\varepsilon)\varphi(k)$ are valid.

The space E is called B_φ*-convex* if there exist $\varepsilon>0$ and $k\geq 2$ such that E is (φ,k,ε)-convex.

Remark. If $\varphi(k)=k^{1/p}$, $1\leq p<2$, then the class of B_φ-convex spaces coincides with that of B_p-convex spaces introduced by A.Shangua in [6], and for $\varphi(k)=k$ - with the class of B-convex spaces introduced by A.Beck (see [4]).

Let us note some simple properties of B_φ-convex spaces. Their proofs differ slightly from these of B_p-convex spaces.

If a normed space E is (φ,k,ε)-convex, then:

1°. for any $\delta:0<\delta<\varepsilon$ E is (φ,k,δ)-convex;

2°. its completion is (φ,k,ε)-convex;

3°. any subspace of E is (φ,k,ε)-convex;

4°. its second dual is (φ,k,ε)-convex;

5°. there exist $p<1$ and $\delta<\varepsilon$ such that for any increasing sequence $\varkappa$ with $\varkappa(n)\geq\varphi^p(n)$ the space E is $(\varkappa,k,\delta)$-convex for all $n\in\mathbb{N}$.

Proof. Notice that the properties 1° and 3° are straightforward.

2°. Let $\overline{x}_1,\dots,\overline{x}_n$ belong to the complement of the space E, i.e., to the Banach space $\overline{E}$, $\|\overline{x}_i\|\leq 1$, $1\leq i\leq k$, $\delta>0$. Since E is dense in $\overline{E}$, there exist $x_1,\dots,x_k\in\mathbb{B}(E)$ such that $\|x_i-\overline{x}_i\|<\delta/k$, $1\leq i\leq k$. Since E is (φ,k,ε)-convex, for certain collection of signs $\pm$ we have that $\|\pm x_1\pm\dots\pm x_k\|\leq(1-\varepsilon)\varphi(k)$. For the same collection we obtain

$$\|\pm x_1\pm\dots\pm x_k\| = \|(\pm x_1\pm\dots\pm x_k)+(\pm(\overline{x}_1-x_1)\pm\dots\pm(\pm(\overline{x}_k-x_k)))\|\leq(1-\varepsilon)\varphi(k)+\delta.$$

By virtue of arbitrariness of $\delta>0$, we have $\inf_{\pm}\|\pm x_1\pm\dots\pm x_k\|\leq(1-\varepsilon)\varphi(k)$.

4°. Let $\mathbb{B}=\mathbb{B}(E)$, $\mathbb{B}^{**}=\mathbb{B}(E^{**})$ and $J:E\to E^{**}$ be the canonic imbedding of E into E^{**}. Then $J(\mathbb{B})$ is dense in $\mathbb{B}^{**}$ in the sense of $*$-weak topology (see [12], Theorem V.5).

Let $x_1^{**},\dots,x_k^{**}$ belong $\mathbb{B}^{**}$, $\delta>0$, and vector $\vec{\alpha}=(\alpha_1,\dots,\alpha_k)$ have the coordinates $\alpha_i\in\{-1,+1\}$, $1\leq i\leq k$. By the norm definition in E^{**}, for any $\vec{\alpha}$ there exists $x_{\vec{\alpha}}^*\in E^*$ with $\|x_{\vec{\alpha}}^*\|=1$ and

$$\|\sum_{i=1}^k\alpha_i x_i^{**}\| \leq |\langle x_{\vec{\alpha}}^*, \sum_{i=1}^k\alpha_i x_i^{**}\rangle|+\delta.$$

Let us consider a neighborhood of zero of E^{**} defined as

$$U_\delta=\bigcap_{\vec{\alpha}}\{x^{**}\in E^{**}:|\langle x_{\vec{\alpha}}^*, x^{**}\rangle|<\delta\},$$

where the intersection is taken by all possible values of $\vec{\alpha}$. Now choose $x_1,\dots,x_k\in E$ such that $J(x_i)\in U_\delta+x_i^{**}$, $1\leq i\leq k$:

$$\|\sum_{i=1}^k\alpha_i x_i^{**}\|\leq|\langle x_{\vec{\alpha}}^*,\sum_{i=1}^k\alpha_i x_i^{**}\rangle|+\delta = |\langle x_{\vec{\alpha}}^*,\sum_{i=1}^k\alpha_i J(x_i)\rangle+\langle x_{\vec{\alpha}}^*,\sum_{i=1}^k\alpha_i(x_i^{**}-J(x_i))\rangle|+\delta\leq\|\sum_{i=1}^k\alpha_i x_i\|+k\delta+\delta.$$

Choosing $\vec{\alpha}$ such that $\|\sum_{i=1}^k\alpha_i x_i\|<(1-\varepsilon)\varphi(k)$, we obtain that $\|\sum_{i=1}^k\alpha_i x_i^{**}\|\leq(1-\varepsilon)\varphi(k)+\delta(k+1)$, or, due to arbitrariness of $\delta>0$, $\inf_{\pm}\|\sum_{i=1}^k\pm x_i^{**}\|\leq(1-\varepsilon)\varphi(k)$.

5°. Set $D(k)=\sup\{\inf_{\pm}\|\sum_{i=1}^k\pm x_i\| \mid x_1,\dots,x_k\in\mathbb{B}(E)\}$. The space E is (φ,k,ε)-convex if and only if $D(k)\leq(1-\varepsilon)\varphi(k)$, or $\varphi(k)\geq D(k)/(1-\varepsilon)$. But then there exist $p<1$ and $\delta<\varepsilon$ such that $\varphi^p(k)\geq D(k)/(1-\delta)$. This implies that for all $\varkappa(k)\geq\varphi^p(k)$ the space E is $(\varkappa,k,\delta)$-convex.

§ 3. Characterization of B_φ-convex spaces via the laws of large numbers for weighted sums

Let E be a Banach space and (X_n) be a sequence of independent centered random elements which are taking values in E. Set

$$T_n = \sum_{k=1}^{k} a_k(n)X_k,$$

where $a=(a_k(n), 1\le k\le n,\ n\in\mathbb{N})$ is triangular array of constants, which will be called a *weight*. Any weight defines the sequences

$$\varphi(n) = 1/\max_{1\le k\le n}|a_k(n)| \quad\text{and}\quad \varkappa(n) = 1/\min_{1\le k\le n}|a_k(n)|,\ n\in\mathbb{N}.$$

In addition, we choose the a so that the sequences $(\varphi(n))_1^\infty$ and $(\varkappa(n))_1^\infty$ be increasing.

We impose the following conditions upon the sequences φ and $\varkappa$ (which are in correspondence with the weight a). Assume that $\chi=(\chi(n))$ is an increasing sequence of positive numbers.

Condition (A). $\chi(nm)\ge\chi(n)\chi(m)$ for all $n,m\in\mathbb{N}$.

Condition (B). There exist $M>1$ and $D>1$ both such that the inequality $\chi(Mn)\le D\chi(n)$ holds true for all $n\in\mathbb{N}$. For a sequence χ we introduce the characteristics:

$$p(\chi) = \inf\Big\{p: \sum_{n=1}^{\infty}\chi^{-p}(n)<\infty\Big\},$$

$$q(\chi) = \sup\{q: \sup_{n\ge2}\chi^{q}(n)/n\le1\}.$$

The next Lemma establishes the relation between these characteristics.

LEMMA 1. *If χ satisfies the condition* (A), *then* $p(\chi)=q(\chi)=\lim\limits_{n\to\infty}\ln n/\ln\chi(n)$. *The supremum in the definition of $q(\chi)$ is to be attained, i.e.,* $\sup\limits_{n\ge2}\chi^{q(\chi)}(n)/n\le1$.

Proof. I. First let us show that the limit $\lim\limits_{n\to\infty}\ln n/\ln\chi(n)$ exists and equals $q(\chi)$. Take an arbitrary $q<q(\chi)$. Then for all $n\ge2$ we have that $\chi(n)\le n^{1/q}$. Hence, $\underline{\lim}\ \ln n/\ln\chi(n)\ge$ $\ge\underline{\lim}\ \ln n/\frac{1}{q}\ln n=q$, i.e., $\underline{\lim}\ \ln n/\ln\chi(n)\ge q(\chi)$.

Now let us take an arbitrary $q>q(\chi)$. Then there exists $N\ge2$ such that $\chi(N)>N^{1/q}$. For all $n\ge N$ we have

$$\chi(n) = \chi(N^{\log_N n}) \ge \chi(N^{[\log_N n]}),$$

where [ ] is the integer part of a number. From this, applying the condition (A) $[\log_N n]$ times, we obtain that

$$\chi(n)\ge\chi(N^{[\log_N n]}(N)\ge\chi^{\log_N n-1}(N)\ge(N^{1/q})^{\log_N n-1}=(n/N)^{1/q}.$$

Consequently,

$$\overline{\lim}\ \ln n/\ln\chi(n) \le \overline{\lim}\ \ln n(\tfrac{1}{q}\ln n+\ln N^{-1/q}) = q,$$

i.e., $\overline{\lim}\ \ln n/\ln\chi(n)\le q(\chi)$. Finally, $\lim\limits_{n\to\infty}\ln n/\ln\chi(n)=q(\chi)$.

II. Now let us turn to the proof of the equality $p(\chi)=q(\chi)$. It is clear that $p(\chi)\ge$ $\ge q(\chi)$, because for any $p>p(\chi)$ the series $\sum\limits_{n=1}^{\infty}\chi^{-p}(n)$ converges. Hence $\chi^{p}(n)/n\longrightarrow\infty$.

Let us show that $p(\chi)\le q(\chi)$. To this end we have to show that for all $p>q(\chi)$ the inequality $p\ge p(\chi)$ is valid. Assume $p>q(\chi)$. For any $\varepsilon>0$ there exists N such that for all $n\ge N$ we have $\ln\chi(n)/\ln n\ge1/q(\chi)-\varepsilon/p$. This implies

$$\sum_{n=N}^{\infty}\chi^{-p}(n) = \sum_{n=N}^{\infty}n^{-p\log_n\chi(n)} = \sum_{n=N}^{\infty}n^{-p\ln\chi(n)/\ln n} \le \sum_{n=N}^{\infty}n^{-p(1/q(\chi)-\varepsilon/p)} = \sum_{n=N}^{\infty}n^{-p/q(\chi)+\varepsilon}.$$

Having chosen $\varepsilon>0$ small enough for $p/q(\chi)-\varepsilon>1$, we obtain the convergence of the series $\sum\limits_{n=1}^{\infty}\chi^{-p}(n)$, i.e., $p\ge p(\chi)$.

III. Finally, let us show that the supremum in definition of $q(\chi)$ is attainable.

Assume the contrary: there exists $N>2$ such that $\chi(N)>N^{1/q(\chi)}$, i.e., $\chi(N)=sN^{1/q(\chi)}$ for some $s>1$. By the condition (A), we have that $\chi(N^k)\ge\chi^k(N)=s^kN^{k/q(\chi)}$. But this yields the contradiction

$$q(\chi)=\lim_{n\to\infty}\ln n/\ln\chi(n)=\lim_{n\to\infty}\ln N^k/\ln\chi(N^k)\le\ln N/(\ln s+(\ln N)/q(\chi))<q(\chi).$$

THEOREM. *Consider the following statements:*

(1) *E is of stable type p, $1\le p<2$ (i.e., is B_p-convex);*

(2) *for all weights a from $\sum_{n=1}^{\infty}P\{\xi>\varphi(n)\}<\infty$ it follows that $T_n\to0$ a.s. for all independent symmetric random elements $(X_k)<\xi$;*

(2') *the statement (2) is fulfilled at least for one weight a;*

(3) *for all weights a the sum $\sum_{k=1}^{n}a_k(n)\varepsilon_kx_k\to0$ in probability with $n\to\infty$, for all $x_k\in\in B(E)$ (ε_k are independent Bernoulli random variables);*

(3') *the statement (3) is fulfilled at least for one weight a;*

(4) *the space E is $B_\varkappa$-convex.*

If φ satisfies the condition (A) and $p(\varphi)\le p$, then (1)⇒(2); if $\varkappa$ satisfies (B), then (3')⇒(4); if $\varkappa$ satisfies (A) and $p\le q(\varkappa)$, then (4)⇒(1); finally, it is evident that

$$\begin{array}{ccc}(2)&\Rightarrow&(2')\\ \Downarrow&&\Downarrow\\ (3)&\Rightarrow&(3').\end{array}$$

In order to prove Theorem we need few Lemmas.

LEMMA 2. *If φ satisfies the condition (A), then for all $p>p(\varphi)$ we have*

$$\sum_{k=n}^{\infty}\varphi^{-p}(k)=O(n\varphi^{-p}(n)).$$

Proof. Indeed, by the condition imposed on φ, one has

$$\sum_{k=n}^{\infty}\varphi^{-p}(k)=\sum_{m=1}^{\infty}\sum_{k=nm}^{n(m+1)}\varphi^{-p}(k)\le\sum_{m=1}^{\infty}n\varphi^{-p}(nm)\le n\varphi^{-p}(n)\sum_{m=1}^{\infty}\varphi^{-p}(m).$$

Note that the latter series converges, because $p>p(\varphi)$.

The following Lemma is a generalization of well-known result (see [13], pp.127-128).

LEMMA 3. *Let $(X_k)<\xi$ and $\sum_{n=1}^{\infty}P\{\xi>\varphi(n)\}<\infty$. If*

$$\sum_{k=n}^{\infty}\varphi^{-p}(k)=O(n\varphi^{-p}(n)),\quad\textit{then}\quad\sum_{n=1}^{\infty}\varphi^{-p}(n)\mathrm{E}\|X_n\|^pI\{\|X_n\|<\varphi(n)\}<\infty.$$

Proof. First note that for $(X_k)<\xi$ we have

$$\mathrm{E}\|X_kI\{\|X_k\|<t\}\|^p\le\mathrm{E}\xi^pI\{\xi<t\}.$$

Thus,

$$\Sigma_1=\sum_{n=1}^{\infty}\varphi^{-p}(n)\mathrm{E}\|X_n\|^pI\{\|X_n\|<\varphi(n)\}\le\sum_{n=1}^{\infty}\varphi^{-p}(n)\mathrm{E}\xi^pI\{\xi<\varphi(n)\}=$$

$$=\sum_{n=1}^{\infty}\varphi^{-p}(n)\sum_{k=1}^{n}\mathrm{E}\xi^pI\{\varphi(k-1)\le\xi<\varphi(k)\}=\sum_{k=1}^{\infty}\mathrm{E}\xi^pI\{\varphi(k-1)\le\xi<\varphi(k)\}\sum_{n=k}^{\infty}\varphi^{-p}(n).$$

In view of the sequence φ be increasing one, we can correctly define an increasing function $\psi(t)$ for which $\psi(\varphi(n))=n$. It is known that $\mathrm{E}\psi(\xi)\le1+\sum_{n=1}^{\infty}P\{\xi>\varphi(n)\}$. Then, due to the condition, $\mathrm{E}\psi(\xi)<\infty$. By the condition imposed on φ, there exists $C>0$ such that

$$\Sigma_1\le\sum_{k=1}^{\infty}\mathrm{E}\psi(\xi)\frac{\xi}{\psi(\xi)}I\{\varphi(k-1)\le\xi<\varphi(k)\}Ck\varphi^{-p}(k)\le C\sum_{k=1}^{\infty}\mathrm{E}\psi(\xi)\frac{\varphi^p(k)}{k-1}I\{\varphi(k-1)\le\xi<\varphi(k)\}k\varphi^{-p}(k)\le$$

$$\le C\sum_{k=1}^{\infty} E\psi(\xi)I\{\varphi(k-1)\le\xi<\varphi(k)\} \le CE\psi(\xi) < \infty.$$

LEMMA 4. *If* (X_k) *are independent symmetric random elements, then* $E(\sup_{n\ge k}\|T_n\|)\le 4E(\sup_{n\ge k}\|S_n\|/\varphi(n))$ *for all* $k\ge 1$.

Proof. By the Levy's inequality (see [14], Proposition V.2.3), for all $k\le m$ we have that $Q=P\{\sup_{k\le n\le m}\|T_n\|>t\}\le 2P\{\|T_n\|>t\}$. But then, by the Kwapien's inequality ([14], Lemma V.4.1(a)), it follows that

$$Q\le 4P\{\|S_m\|/\varphi(m)>t\}\le 4P\{\sup_{k\le n\le m}\|S_n\|/\varphi(n)>t\}.$$

Consequently, $P\{\sup_{n\ge k}\|T_n\|>t\}\le 4P\{\sup_{n\ge k}\|S_n\|/\varphi(n)>t\}$. Hence,

$$E(\sup_{n\ge k}\|T_n\|)=\int_0^\infty P\{\sup_{n\ge k}\|T_n\|>t\}dt\le 4\int_0^\infty P\{\sup_{n\ge k}\|S_n\|/\varphi(n)>t\}dt=4E(\sup_{n\ge k}\|S_n\|/\varphi(n)).$$

LEMMA 5. *Let* E *possess the Rademacher type* p, $1<p<2$. *If* (X_k) *are independent symmetric random elements satisfying* $\sum_{k=1}^{\infty}E\|X_k\|^p/\varphi^p(k)<\infty$, *then* $E(\sup_{n\ge1}\|S_n\|/\varphi(n))<\infty$.

Proof. Let us set $m_k=\min\{n:\varphi(n)\ge 2^k\}$ and $N_k=\{n:m_k\le n<m_{k+1}\}$ for any $k\ge0$. Then, by the choice of N_k and by Levy's inequality, we have that

$$Q=P\{\sup_{n\ge1}\|S_n\|/\varphi(n)>t\}=P\{\sup_{k\ge0}\max_{n\in N_k}\|S_n\|/\varphi(n)>t\}\le$$

$$\le\sum_{k=0}^{\infty}P\{\max_{n\in N_k}\|S_n\|/\varphi(n)>t\}\le\sum_{k=0}^{\infty}P\{\max_{n\in N_k}\|S_n\|>2^kt\}\le 2\sum_{k=0}^{\infty}P\{\|S_{m_{k+1}-1}\|>2^kt\}.$$

Now, using Chebyshev's inequality and taking into account that E is of the Rademacher type p (see [11], ch.2, §2; [2], [3], ch.3-4), we obtain that

$$Q\le 8t^{-p}\sum_{k=0}^{\infty}2^{-kp}E\Big\|\sum_{i=1}^{m_{k+1}-1}X_i\Big\|^p\le Ct^{-p}\sum_{k=0}^{\infty}2^{-kp}\sum_{i=1}^{m_{k+1}-1}E\|X_i\|^p.$$

Altering the order of summing, we obtain

$$Q\le Ct^{-p}\sum_{i=1}^{\infty}E\|X_i\|^p\sum_{k\ge k_0}2^{-kp},$$

where $k_0=\min\{k:m_{k+1}-1\ge i\}$. Note that the for k_0 the inequalities $m_{k_0+1}-1\ge i$ and $\varphi(m_{k_0+1}-1)>2^{k_0+1}$ are valid. Let us estimate the sum:

$$\sum_{k\ge k_0}2^{-kp}=\frac{2^p}{1-2^{-p}}2^{-(k_0+1)p}\le C(\varphi(m_{k_0+1}-1))^{-p}\le C\varphi^{-p}(i).$$

Hence,

$$Q\le Ct^{-p}\sum_{i=1}^{\infty}E\|X_i\|^p/\varphi^p(i).$$

Then

$$E\sup_{n\ge1}\|S_n\|/\varphi(n)\le 1+\int_1^\infty P\{\sup_{n\ge1}\|S_n\|/\varphi(n)>t\}dt\le 1+C\sum_{i=1}^{\infty}E\|X_i\|^p/\varphi^p(i)\int_1^\infty t^{-p}dt<\infty.$$

The following Lemma can be considered as an analog of Chung's law of large numbers (cf. [15]) for weighted sums.

LEMMA 6. *If the space* E *is of Rademacher type* p, $1<p<2$, *then for any independent symmetric random elements* (X_k) *the condition* $\sum_{i=1}^{\infty}E\|X_k\|^p/\varphi^p(k)<\infty$ *implies* $T_n\to0$ *a.s.*

Proof. As the space E is of the Rademacher type p (see [1], ch.2, §2; [2]; [3], ch. 3-4), so

$$\mathbb{E}\Big\| \sum_{k=1}^{m} X_k/\varphi(k)\Big\|^p \le C\sum_{k=1}^{m}\mathbb{E}\|X_k\|^p/\varphi^p(k).$$

Therefore, the series $\sum_{k=1}^{\infty} X_k/\varphi(k)$ converges in $L^p(E)$. But then ([14], Theorem V.2.3) it converges a.s. By the Kronecker's Lemma, we have $S_n/\varphi(n)\to 0$ a.s. It means that $\sup_{k\ge n} S_k/\varphi(k)\to 0$ a.s. with $n\to\infty$. By Lemma 5 we have that $\mathbb{E}\sup_{k\ge 1}\|S_n\|/\varphi(n)<\infty$. By the Lebesgue Theorem, $\mathbb{E}\sup_{k\ge n}\|S_k\|/\varphi(k)\to 0$ with $n\to\infty$. By Lemma 4, $\mathbb{E}\sup_{k\ge n}\|T_k\|\to 0$ with $n\to\infty$. Finally, by the Chebyshev's inequality, we have that $\mathbb{P}\{\sup_{k\ge n}\|T_k\|>\varepsilon\}\to 0$ with $n\to\infty$ for all $\varepsilon>0$. Hence $T_n\to 0$ a.s.

LEMMA 7. *Let χ satisfy the condition* (B) *and the space E be non-B_χ-convex. Then there exist the sequences $(x_n)\in\mathbb{B}(E)$ and $(n_k)_{k=1}^{\infty}$ such that*

$$\inf_{\pm}\Big\| \frac{1}{\chi(n_k)}\sum_{i=1}^{n_k}\pm x_i\Big\|>1/(2D).$$

We will prove Lemma by the induction. Put $n_1=1$ and take in the capacity of x_1 an element E with $1/(2D)<\|x_1\|\le 1$.

Assume that there are $n_1<n_2<\dots<n_m$ and $(x_i)_{i=1}^{n_m}\subset\mathbb{B}(E)$ which satisfy the conditions of Lemma. Choose n_{m+1} so that the following inequalities hold true: $n_{m+1}>Mn_m/(M-1)$ (i.e., $n_{m+1}<M(n_{m+1}-n_m)$) and $\chi(n_{m+1})>4Dn_m$.

As the space E is not B_χ-convex, so there exist $x_{n_m+1},\dots,x_{n_{m+1}}\in\mathbb{B}(E)$ such that for arbitrary chosen signs "±" one has

$$\|\pm x_{n_m+1}\pm\dots\pm x_{n_{m+1}}\| > (1-1/4)\chi(n_{m+1}-n_m).$$

But then, by virtue of the condition (B), it follows

$$\frac{1}{\chi(n_{m+1})}\|\pm x_1\pm\dots\pm x_{n_{m+1}}\| \ge \frac{1}{\chi(n_{m+1})}\|\pm x_{n_m+1}\pm\dots\pm x_{n_{m+1}}\|-\frac{1}{\chi(n_{m+1})}\|\pm x_1\pm\dots\pm x_{n_m}\| \ge$$
$$\ge \frac{1}{\chi(M(n_{m+1}-n_m))}\|\pm x_{n_m+1}\pm\dots\pm x_{n_{m+1}}\| - n_m/\chi(n_{m+1}) \ge$$
$$\ge (1-1/4)\chi(n_{m+1}-n_m)/\chi(M(n_{m+1}-n_m))-1/(4D) \ge 1/(2D).$$

Proof of Theorem. First let us show that (1)⇒(2). As E is of stable type $p<2$ and $p(\varphi)\le p$, there exists $q>p(\varphi)$ such that E is of the Rademacher type q (see [1], ch.2, §2; [2]; [3], ch.3-4). Let $X_n'=X_nI\{\|X_n\|\le\varphi(n)\}$. Then, by virtue of Lemmas 2 and 3, the series $\sum_{n=1}^{\infty}\varphi^{-q}(n)\mathbb{E}\|X_n'\|^q<\infty$. From this, by Lemma 6, we have $\sum_{k=1}^{n}a_k(n)X_k'\to 0$ a.s. Moreover,

$$\sum_{n=1}^{\infty}P\{X_n\ne X_n'\} = \sum_{n=1}^{\infty}P\{X_n>\varphi(n)\} \le \sum_{n=1}^{\infty}P\{\xi>\varphi(n)\} < \infty.$$

By the Borel-Cantelli's Lemma,

$$P\{X_n\ne X_n' \text{ infinite times}\} = 0.$$

Therefore, $T_n\to 0$ a.s. with $n\to\infty$.

Now let us demonstrate that (3')⇒(4). Let E be non-B_χ-convex. Then for a sequence $(x_n)\subset\mathbb{B}(E)$ the assertion of Lemma 7 takes place. By the condition,

$$Y_n = \sum_{k=1}^{n}a_k(n)\varepsilon_k x_k\to 0$$

in probability with $n\to\infty$. We fix $\varepsilon>0$ and choose n large enough for $P\{\|Y_n\|>\varepsilon\}<1/8$. Put $\eta_k=a_k(n)\varepsilon_k x_k$. Then $\|\eta_k\|\le a_k(n)\le 1/\varphi(n)$ and $Y_n=\sum_{k=1}^{n}\eta_k$. By virtue of Levy's and Kolmogorov's inequalities (see [14], pp.211,223), we have

$$\frac{1}{8} \geq P\left\{\left\| \sum_{k=1}^{n} \eta_k \right\| \geq \varepsilon\right\} \geq \frac{1}{4}\left(1 - \frac{(\varepsilon + 1/\varphi(n))^2 + \varepsilon^2/2}{\mathbb{E}\|Y_n\|^2}\right).$$

Therefore, $\mathbb{E}\|Y_n\|^2 \leq 2((\varepsilon+1/\varphi(n))^2+\varepsilon^2/2)$. As ε is arbitrary, so $\mathbb{E}\|Y_n\|^2 \to 0$ with $n\to\infty$.

Note that due to Lemma V.4.1 (b) (see [14]) it follows that

$$\mathbb{E}\left\|\frac{1}{\varkappa(n)} \sum_{k=1}^{n} \varepsilon_k x_k\right\|^2 \leq 2\mathbb{E}\left\|\sum_{k=1}^{n} a_k(n)\varepsilon_k x_k\right\|^2 = \mathbb{E}\|Y_n\|^2 \to 0.$$

i.e., $\frac{1}{\varkappa(n)} \sum_{k=1}^{n} \varepsilon_k x_k \to 0$ in $L^2(E)$ and, consequently, "in probability". But this contradicts the statement of Lemma.

Finally, let us show that (4)⇒(1). Suppose that E is being not of the stable type $p=q(\varkappa)$. Then l_p is finitely representable in E (see [1], ch.2, §2; [2]; [3], ch.3-4), i.e., for all $n\in\mathbb{N}$ and $\varepsilon>0$ there exist $x_1,\dots,x_n\in E$ such that for all $s_1,\dots,s_n\in\mathbb{R}$ the inequality

$$(1-\varepsilon)\left(\sum_{k=1}^{n}|s_k|^p\right)^{1/p} \leq \left\|\sum_{k=1}^{n} s_k x_k\right\| \leq \left(\sum_{k=1}^{n}|s_k|^p\right)^{1/p}$$

is valid.

As E is $B_\varkappa$-convex, so there exist $n\geq 2$ and $\varepsilon>0$ such that for all $x_k\in\mathbb{B}(E)$ we have that $\|\pm x_1\pm\dots\pm x_n\| \leq (1-\varepsilon)\varkappa(n)$. Taking namely this collection of signs in the capacity of s_i, we obtain that

$$n^{1/p}(1-\varepsilon/2) \leq \left\|\sum_{k=1}^{n} \pm x_k\right\| \leq (1-\varepsilon)\varkappa(n),$$

or

$$\varkappa^p(n)/n \geq ((1-\varepsilon/2)/(1-\varepsilon))^p > 1.$$

But this contradicts the choice of $p=q(\varkappa)$, because due to Lemma 1 we have that

$$\sup_{n\geq 2} \varkappa^{q(\varkappa)}(n)/n \leq 1.$$

§ 4. Subsequent properties of B_φ-convex spaces

With the help of proved Theorem we can obtain a number of properties of B_φ-convex spaces, which are not so trivial as 1°-5° were.

6°. *If φ satisfies (A) and (B), then the following statements are equivalent:*

(1) *the space E is of stable type $p=p(\varphi)$;*

(2) *for any independent centered random elements (X_k) such that $(X_k)<\xi$ and $\sum_{n=1}^{\infty} P\{\xi>\varphi(n)\}<\infty$, the sequence $S_n/\varphi(n)\to 0$ a.s. with $n\to\infty$;*

(3) *the space E is B_φ-convex.*

Remark. It is obvious that power functions satisfy both the conditions (A) and (B). In this case, we obtain the statements of Marcus-Woyczynski-Shangua's Theorem. Therefore, the example $\varphi(n)=n^{1/p}(\ln n+1)$, $1<p<2$, must be more interesting one.

Note that the implication (4)⇒(1) of Theorem implies the next statement: the notion of B_φ-convexity (φ satisfies (A)) makes sense only for $p(\varphi)<2$, because the notion of

7°. *Let φ satisfy the condition* (A). *The space E then is B_φ-convex if and only if $A_\varphi(k)\longrightarrow 0$ with $k\longrightarrow\infty$.*

Proof. If $A_\varphi(k)\longrightarrow 0$, then, obviously, the space E is B_φ-convex.

Vice versa, if E is B_φ-convex, then, by implication (4)⇒(1) of Theorem, E is of the stable type $p>q(\varphi)$. For any $x_1,\dots,x_k\in\mathbb{B}(E)$ the series is such that

$$\sum_{k=1}^{\infty}\|x_k\|^p/\varphi^p(k)\leqslant\sum_{k=1}^{\infty}\varphi^{-p}(k).$$

Consequently, it converges, because $p>p(\varphi)$. By the Kronecker's Lemma,

$$\frac{1}{\varphi^p(k)}\sum_{i=1}^{k}\|x_i\|^p\longrightarrow 0 \text{ with } k\longrightarrow\infty.$$

Since E is of Rademacher type p, we have that

$$\left(\mathbb{E}\,\frac{1}{\varphi(k)}\left\|\sum_{i=1}^{k}\varepsilon_i x_i\right\|^p\right)^{1/p}\leqslant C\left(\frac{1}{\varphi^p(k)}\sum_{i=1}^{k}\|x_k\|^p\right)^{1/p}\longrightarrow 0$$

with $k\longrightarrow\infty$.

With the help of the property 7° it is easy to establish

8°. *Let E_1 and E_2 be normed spaces, $R:E_1\longrightarrow E_2$ be linear continuous and opened operator. Moreover, assume that E_1 is B_φ-convex and φ satisfies* (A). *Then $R(E_1)\subseteq E_2$ is B_φ-convex.*

Proof. Since R is continuous, there exists $L>0$ such that $\|R(x)\|\leqslant L$ for all $x\in\mathbb{B}=\mathbb{B}(E_1)$. The openness of the mapping implies that $R(\mathbb{B})$ is the neighborhood of zero in $R(E_1)$, i.e., there exists $\delta>0$, for which we have $\{y\in R(E_1):\|y\|<\delta\}\subset R(\mathbb{B})$.

Let us choose k so large that $A_\varphi(k)\leqslant\delta/2L$. This can be done due to the property 7°. Let $y_1,\dots,y_k$ belong to the unit ball of the space $R(E_1)$. Then we have that $\delta y_1,\dots,\delta y_k\in R(\mathbb{B})$, i.e., there exist $x_1,\dots,x_k\in\mathbb{B}$ such that $Rx_i=\delta y_i$, $1\leqslant i\leqslant k$. Let us arrange the signs in such an order that the inequality

$$\frac{1}{\varphi(k)}\left\|\sum_{i=1}^{k}\pm x_i\right\|\leqslant\delta/(2L)$$

holds true. Then

$$\frac{1}{\varphi(k)}\left\|\sum_{i=1}^{k}\pm y_i\right\|=\frac{1}{\delta\varphi(k)}\left\|\sum_{i=1}^{k}\pm\delta y_i\right\|=\frac{1}{\delta\varphi(k)}\left\|\sum_{i=1}^{k}\pm Rx_i\right\|\leqslant\frac{1}{\delta\varphi(k)}\left\|\sum_{i=1}^{k}\pm x_i\right\|\leqslant 1/2.$$

Hence, $R(E_1)$ is $(\varphi,k,1/2)$-convex.

Corollary. If the normed spaces E_1 and E_2 are isomorphic and φ satisfies (A), then E_1 is B_φ-convex if and only if E_2 is B_φ-convex.

I wish to express gratitude to A.N.Chuprunov for his valuable advice.

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11 April 1991 Kazan