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**ON COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE
NEGATIVELY ASSOCIATED RANDOM VARIABLES¹⁾**

С помощью методов, развитых в [4], мы устанавливаем результат о полной сходимости в схеме серий построчно отрицательно ассоциированных случайных величин, который обобщает результаты работ [2], [1] и [8].

Ключевые слова и фразы: отрицательно ассоциированные случайные величины, схема серий построчно отрицательно ассоциированных случайных величин, полная сходимость.

The paper [4] provides sufficient conditions for complete convergence for sums of arrays of rowwise independent random variables. The wide majority of results on complete convergence known in literature could be derived from the main result of [4]. In that paper one can find also a short history of the complete convergence notion and the corresponding references. The purpose of the present investigation is to prove by the methods developed in [4] that the main results of [2], [1], and [8] remain true for negatively associated random variables.

In this paper we assume that $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ is an array of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $\{k_n, n \geq 1\}$ is a sequence of positive integers such that $\lim_{n \rightarrow \infty} k_n = \infty$, and $\{c_n, n \geq 1\}$ is a sequence of positive constants. For an event $A \in \mathcal{F}$ we denote by $I\{A\}$ the indicator function. We note that all the results of this paper remain true in the case $k_n = \infty$ for some/all $n \geq 1$, provided the series $\sum_{k=1}^{\infty} X_{nk}$ converges almost surely. Certainly, we should consider sup instead of max in the case of infinite sums. In what follows, C always stands for a positive constant which may differ from one place to another.

Recall that a finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be *negatively associated* (abbreviated to NA in what follows) if for any disjoint subsets A and B of $\{1, 2, \dots, n\}$ and any real coordinate-wise nondecreasing scalar functions f on \mathbf{R}^A and g on \mathbf{R}^B ,

$$\text{cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$$

whenever the covariance exists. An infinite family of random variables $\{X_i, i \geq 1\}$ is NA if every finite subfamily is NA.

This concept was introduced in [3]. The authors of [3] also pointed out and proved in their paper that a number of well-known multivariate distributions possess the NA property. The following exponential inequality of Kolmogorov's type is proved in [7].

Lemma 1. *Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of NA mean zero random variables with $\mathbf{E}|X_i|^2 < \infty$ for every $1 \leq i \leq n$ and let*

$$B_n = \sum_{i=1}^n \mathbf{E}X_i^2.$$

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Then for all $\varepsilon > 0$ and $a > 0$

$$\mathbf{P}\left\{\max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right| \geq \varepsilon\right\} \leq 2\mathbf{P}\left\{\max_{1 \leq i \leq n} |X_i| > a\right\} + 4 \exp\left\{-\frac{\varepsilon^2}{4(a\varepsilon + B_n)} \left[1 + \frac{2}{3} \ln\left(1 + \frac{a\varepsilon}{B_n}\right)\right]\right\}.$$

Our main result, which could be considered as a partial generalization of Theorem 1 in [4], is the following.

Theorem 1. Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise NA random variables. Suppose that for every $\varepsilon > 0$ and some $\delta > 0$

- (i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} \mathbf{P}\{|X_{ni}| > \varepsilon\} < \infty$,
- (ii) there exists $j \geq 1$ such that $\sum_{n=1}^{\infty} c_n (\sum_{i=1}^{k_n} \mathbf{D}X_{ni} I\{|X_{ni}| \leq \delta\})^j < \infty$. Then

$$\sum_{n=1}^{\infty} c_n \mathbf{P}\left\{\max_{1 \leq m \leq k_n} \left| \sum_{k=1}^m [X_{nk} - \mathbf{E}X_{nk} I\{|X_{nk}| \leq \delta\}] \right| > \varepsilon\right\} < \infty$$

for any $\varepsilon > 0$.

P r o o f. Let $\{Y_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be a *monotone truncation* of $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$, namely, $Y_{nk} = X_{nk} I\{|X_{nk}| \leq \delta\} + \delta I\{X_{nk} > \delta\} - \delta I\{X_{nk} < -\delta\}$, where $\delta > 0$ and $1 \leq k \leq k_n, n \geq 1$. Note that by Property 6 in [3] (applied twice) we can conclude that $\{Y_{nk} - \mathbf{E}Y_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ is an array of rowwise NA random variables.

For $n \geq 1$ and $1 \leq m \leq k_n$ let

$$\begin{aligned} T_m &= \sum_{k=1}^m Y_{nk}, & S_m &= \sum_{k=1}^m X_{nk}, \\ S'_m &= \sum_{k=1}^m X_{nk} I\{|X_{nk}| \leq \delta\}, & Z_m &= \sum_{k=1}^m (\delta I\{X_{nk} > \delta\} - \delta I\{X_{nk} < -\delta\}), \\ A &= \bigcap_{k=1}^{k_n} \{X_{nk} = X_{nk} I\{|X_{nk}| \leq \delta\}\}, & A^c &= \bigcup_{k=1}^{k_n} \{X_{nk} \neq X_{nk} I\{|X_{nk}| \leq \delta\}\}. \end{aligned}$$

Note that for any $n \geq 1$

$$\begin{aligned} \mathbf{P}\left\{\max_{1 \leq m \leq k_n} |S_m - \mathbf{E}S'_m| > \varepsilon\right\} &= \mathbf{P}\left\{\left\{\max_{1 \leq m \leq k_n} |S_m - \mathbf{E}S'_m| > \varepsilon\right\} \cap A^c\right\} \\ &\quad + \mathbf{P}\left\{\left\{\max_{1 \leq m \leq k_n} |S'_m - \mathbf{E}S'_m| > \varepsilon\right\} \cap A\right\} \\ &\leq \sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\} + \mathbf{P}\left\{\max_{1 \leq m \leq k_n} |S'_m - \mathbf{E}S'_m| > \varepsilon\right\}. \end{aligned}$$

The last term can be estimated in the following way:

$$\begin{aligned} \mathbf{P}\left\{\max_{1 \leq m \leq k_n} |S'_m - \mathbf{E}S'_m| > \varepsilon\right\} &= \mathbf{P}\left\{\max_{1 \leq m \leq k_n} |(T_m - \mathbf{E}T_m) + (Z_m - \mathbf{E}Z_m)| > \varepsilon\right\} \\ &\leq \mathbf{P}\left\{\max_{1 \leq m \leq k_n} |T_m - \mathbf{E}T_m| > \frac{\varepsilon}{2}\right\} + \mathbf{P}\left\{\max_{1 \leq m \leq k_n} |Z_m - \mathbf{E}Z_m| > \frac{\varepsilon}{2}\right\}. \end{aligned}$$

By the Markov inequality $\mathbf{P}\{\max_{1 \leq m \leq k_n} |Z_m - \mathbf{E}Z_m| > \varepsilon/2\} \leq C \sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\}$. By (i) it is enough to prove that

$$\sum_{n=1}^{\infty} c_n \mathbf{P}\left\{\max_{1 \leq m \leq k_n} |T_m - \mathbf{E}T_m| > \varepsilon\right\} < \infty.$$

Let $B_n = \sum_{k=1}^{k_n} \mathbf{D}(Y_{nk})$. For any $\varepsilon > 0$ and $a > 0$ set $d = \min\{1, a\varepsilon/(18\delta^2), a/(12\delta)\}$,

$$\begin{aligned} \mathbf{N}_1 &= \{n: B_n > a\varepsilon\}, \quad \mathbf{N}_2 = \left\{n: \sum_{k=1}^{k_n} \mathbf{P}\left\{|X_{nk}| > \min\left\{\delta, \frac{a}{3}\right\}\right\} > d\right\}, \\ \mathbf{N}_3 &= \left\{n: \sum_{k=1}^{k_n} \mathbf{D}X_{nk} I\{|X_{nk}| \leq \delta\} > \frac{a\varepsilon}{2}\right\}, \quad \mathbf{N}_4 = \mathbf{N} - (\mathbf{N}_2 \cup \mathbf{N}_3). \end{aligned}$$

We can state that $\mathbf{N}_1 \subseteq \mathbf{N}_2 \cup \mathbf{N}_3$, because

$$\begin{aligned} B_n &= \sum_{k=1}^{k_n} \left[\mathbf{D}X_{nk} I\{|X_{nk}| \leq \delta\} + \mathbf{D}(\delta I\{X_{nk} > \delta\} - \delta I\{X_{nk} < -\delta\}) \right. \\ &\quad \left. + 2 \operatorname{cov}(X_{nk} I\{|X_{nk}| \leq \delta\}, \delta I\{X_{nk} > \delta\} - \delta I\{X_{nk} < -\delta\}) \right] \\ &\leq \sum_{k=1}^{k_n} \mathbf{D}X_{nk} I\{|X_{nk}| \leq \delta\} + 9\delta^2 \sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\}. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{n \in \mathbf{N}_2 \cup \mathbf{N}_3} c_n \mathbf{P}\left\{\max_{1 \leq m \leq k_n} |T_m - \mathbf{E}T_m| > \varepsilon\right\} &\leq \sum_{n \in \mathbf{N}_2 \cup \mathbf{N}_3} c_n \\ &\leq \frac{1}{d} \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\} + \frac{2^j}{(a\varepsilon)^j} \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} \mathbf{D}X_{nk} I\{|X_{nk}| \leq \delta\} \right)^j < \infty. \end{aligned}$$

Hence it suffices to prove that $\sum_{n \in \mathbf{N}_4} c_n \mathbf{P}\{\max_{1 \leq m \leq k_n} |\sum_{k=1}^m (Y_{nk} - \mathbf{E}Y_{nk})| > \varepsilon\} < \infty$. By Lemma 1 we have

$$\begin{aligned} \sum_{n \in \mathbf{N}_4} c_n \mathbf{P}\left\{\max_{1 \leq m \leq k_n} \left| \sum_{k=1}^m (Y_{nk} - \mathbf{E}Y_{nk}) \right| > \varepsilon\right\} \\ \leq \sum_{n \in \mathbf{N}_4} c_n \left[2\mathbf{P}\left\{\max_{1 \leq k \leq k_n} |Y_{nk} - \mathbf{E}Y_{nk}| > a\right\} \right. \\ \left. + 4 \exp\left\{-\frac{\varepsilon^2}{4(a\varepsilon + B_n)} \left[1 + \frac{2}{3} \ln\left(1 + \frac{a\varepsilon}{B_n}\right)\right]\right\} \right]. \end{aligned}$$

Note that for any $n \in \mathbf{N}_4$

$$\begin{aligned} \max_{1 \leq k \leq k_n} |\mathbf{E}Y_{nk}| &\leq \max_{1 \leq k \leq k_n} \mathbf{E}|Y_{nk}| \\ &\leq \max_{1 \leq k \leq k_n} \left(\delta \mathbf{P}\{|X_{nk}| > \delta\} + \mathbf{E}|X_{nk}| I\left\{|X_{nk}| \leq \frac{a}{3}\right\} + \mathbf{E}|X_{nk}| I\left\{\frac{a}{3} < |X_{nk}| \leq \delta\right\} \right) \\ &\leq 2\delta \sum_{k=1}^{k_n} \mathbf{P}\left\{|X_{nk}| > \min\left\{\frac{a}{3}, \delta\right\}\right\} + \frac{a}{3} \leq 2\delta d + \frac{a}{3} \leq \frac{a}{2} \end{aligned}$$

(the last but one inequality holds, because $n \notin \mathbf{N}_2$). This implies that, for any $n \in \mathbf{N}_4$, $\max_{1 \leq k \leq k_n} |\mathbf{E}Y_{nk}| \leq a/2$ and

$$\begin{aligned} \sum_{n \in \mathbf{N}_4} c_n \mathbf{P}\left\{\max_{1 \leq k \leq k_n} |Y_{nk} - \mathbf{E}Y_{nk}| > a\right\} &\leq \sum_{n=1}^{\infty} c_n \mathbf{P}\left\{\max_{1 \leq k \leq k_n} |Y_{nk}| > \frac{a}{2}\right\} \\ &\leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbf{P}\left\{|X_{nk}| > \min\left\{\delta, \frac{a}{2}\right\}\right\} < \infty \quad (\text{by (i)}). \end{aligned}$$

Therefore it is sufficient to prove that

$$A = \sum_{n \in \mathbf{N}_4} c_n \exp \left\{ -\frac{\varepsilon^2}{4(a\varepsilon + B_n)} \left[1 + \frac{2}{3} \ln \left(1 + \frac{a\varepsilon}{B_n} \right) \right] \right\} < \infty.$$

When $n \in \mathbf{N}_4$, $B_n \leq a\varepsilon$, and $\sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\} \leq 1$. Let $a = \varepsilon/(12j)$. We have

$$\begin{aligned} A &\leq \exp \left\{ -\frac{3}{2}j \right\} \sum_{n \in \mathbf{N}_4} c_n \exp \left\{ -j \ln \frac{B_n + a\varepsilon}{B_n} \right\} \\ &= C \sum_{n \in \mathbf{N}_4} c_n \left(\frac{B_n}{B_n + a\varepsilon} \right)^j \leq C \sum_{n \in \mathbf{N}_4} c_n \left(\frac{B_n}{a\varepsilon} \right)^j = C \sum_{n \in \mathbf{N}_4} c_n (B_n)^j \\ &\leq C \sum_{n \in \mathbf{N}_4} c_n \left[9\delta^2 \sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\} + \sum_{k=1}^{k_n} \mathbf{D}X_{nk} I\{|X_{nk}| \leq \delta\} \right]^j \\ &\leq C \sum_{n \in \mathbf{N}_4} c_n \left\{ (9\delta^2)^j \left[\sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\} \right]^j + \left[\sum_{k=1}^{k_n} \mathbf{D}X_{nk} I\{|X_{nk}| \leq \delta\} \right]^j \right\} \\ &\leq C \sum_{n \in \mathbf{N}_4} c_n \left\{ (9\delta^2)^j \sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\} + \left[\sum_{k=1}^{k_n} \mathbf{D}X_{nk} I\{|X_{nk}| \leq \delta\} \right]^j \right\} < \infty \end{aligned}$$

by the assumptions. The proof is completed.

In the following two corollaries of Theorem 1 we assume that $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ is an array of rowwise NA random variables such that conditions (i) and (ii) are satisfied. Corollary 1 shows that the main result of [2] and [8] remains true for negative associated random variables.

Corollary 1. *If $\sum_{i=1}^{k_n} \mathbf{E}X_{ni} I\{|X_{ni}| \leq \delta\} \rightarrow 0$ as $n \rightarrow \infty$, then*

$$\sum_{n=1}^{\infty} c_n \mathbf{P} \left\{ \left| \sum_{k=1}^{k_n} X_{nk} \right| > \varepsilon \right\} < \infty \quad \text{for any } \varepsilon > 0.$$

Corollary 2 shows that the main result of [1] remains true for negatively associated random variables.

Corollary 2. *If $\max_{1 \leq m \leq k_n} |\sum_{i=1}^m \mathbf{E}X_{ni} I\{|X_{ni}| \leq \delta\}| \rightarrow 0$ as $n \rightarrow \infty$, then $\sum_{n=1}^{\infty} c_n \mathbf{P}\{\max_{1 \leq m \leq k_n} |\sum_{k=1}^m X_{nk}| > \varepsilon\} < \infty$ for any $\varepsilon > 0$.*

Similar argument as in [2, Remark 2] gives the following result.

Corollary 3. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise NA random variables with $\mathbf{E}X_{nk} = 0$ for all $n \geq 1$ and $1 \leq k \leq k_n$. Let $\phi(x)$ be a real function such that for some $\delta > 0$*

$$\sup_{x > \delta} \frac{x}{\phi(x)} < \infty \quad \text{and} \quad \sup_{0 \leq x \leq \delta} \frac{x^2}{\phi(x)} < \infty.$$

Suppose that for all $\varepsilon > 0$, conditions (i) and

(ii)' there exists $j \geq 1$ such that $\sum_{n=1}^{\infty} c_n (\sum_{k=1}^{k_n} \mathbf{E}\phi(|X_{nk}|))^j < \infty$ and $\sum_{k=1}^{k_n} \mathbf{E}\phi(|X_{nk}|) \rightarrow 0$ as $n \rightarrow \infty$ are satisfied. Then

$$\sum_{n=1}^{\infty} c_n \mathbf{P} \left\{ \max_{1 \leq m \leq k_n} \left| \sum_{k=1}^m X_{nk} \right| > \varepsilon \right\} < \infty$$

for any $\varepsilon > 0$.

P r o o f. We show that the conditions of Corollary 2 are satisfied.

For (ii) note that

$$\sum_{k=1}^{k_n} \mathbf{E}X_{nk}^2 I\{|X_{nk}| \leq \delta\} \leq \sup_{0 \leq x \leq \delta} \frac{x^2}{\phi(x)} \sum_{k=1}^{k_n} \mathbf{E}\phi(|X_{nk}|).$$

Since $\mathbf{E}X_{nk} = 0$ it follows that

$$\begin{aligned} \max_{1 \leq m \leq k_n} \left| \sum_{k=1}^m \mathbf{E}X_{nk} I\{|X_{nk}| \leq \delta\} \right| &= \max_{1 \leq m \leq k_n} \left| \sum_{k=1}^m \mathbf{E}X_{nk} I\{|X_{nk}| > \delta\} \right| \\ &\leq \sup_{x > \delta} \frac{x}{\phi(x)} \sum_{k=1}^{k_n} \mathbf{E}\phi(|X_{nk}|) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

R e m a r k. By Theorem 1 and Corollary 3 we have that Corollaries 1 and 2 in [2] are also true if we assume negative association instead of independence and replace weighted sums by the maxima of weighted sums. By Theorem 1, it is easy to obtain the sufficiency parts of Theorems 2.1 and 2.3 in [6] and Theorems 2.1 and 2.2 in [5].

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REFERENCES

1. *Hu T.-C., Ordóñez Cabrera M., Sung S.H., Volodin A.* Complete convergence for arrays of rowwise independent random variables. — *Commun. Korean Math. Soc.*, 2003, v. 18, № 2, p. 375–383.
2. *Hu T.-C., Szynal D., Volodin A.I.* A note on complete convergence for arrays. — *Statist. Probab. Lett.*, 1998, v. 38, № 1, p. 27–31.
3. *Joag-Dev K., Proschan F.* Negative association of random variables, with applications. — *Ann. Statist.*, 1983, v. 11, № 1, p. 286–295.
4. *Kruglov V.M., Volodin A.I., Hu T.-C.* On complete convergence for arrays. — *Statist. Probab. Lett.*, 2006, v. 76, № 15, p. 1631–1640.
5. *Liang H. Y.* Complete convergence for weighted sums of negatively associated random variables. — *Statist. Probab. Lett.*, 2000, v. 48, № 4, p. 317–325.
6. *Liang H. Y., Su C.* Complete convergence for weighted sums of NA sequences. — *Statist. Probab. Lett.*, 1999, v. 45, № 1, p. 85–95.
7. *Shao Q. M.* A comparison theorem on moment inequalities between negatively associated and independent random variables. — *J. Theoret. Probab.*, 2000, v. 13, № 2, p. 343–356.
8. *Sung S.H., Volodin A.I., Hu T.-C.* More on complete convergence for arrays. — *Statist. Probab. Lett.*, 2005, v. 71, № 4, p. 303–311.

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